Introduction to fusion categories

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Les Houches, April, 2024

Why categories?

- Original framework for mathematical constructions of topological field theories in 2+1 dimensions (Reshetikhin-Turaev (1991), Turaev-Viro (1992)).
- Explicit lattice Hamiltonian formulations (Levin-Wen (2005)).
- Extensions to higher dimensions (next week lecture by C. Delcamp).
- Generalized symmetries and dualities (next week lecture by L. Lootens).
- Provides many new models (general constructions), and also helps to understand why things work: many calculations are replaced by drawings!

Various constructions of TQFT's in 2+1 dimensions

- Reshetikhin-Turaev construction requires a modular tensor category $\mathcal{C}$. Defines a Hilbert space $Z_{\mathrm{RT},\mathcal{C}}(\Sigma)$ for any closed surface Σ and a vector $Z_{\mathrm{RT},\mathcal{C}}(\mathcal{M}) \in Z_{\mathrm{RT},\mathcal{C}}(\partial\mathcal{M})$ for any smooth 3-manifold $\mathcal{M}$. This is a non local construction, since it uses surgery of manifolds.
- Turaev-Viro (generalized by Barrett-Westbury (1996)) requires a (spherical) fusion category $\mathcal{A}$ as input. This construction is local, and it involves discretized path integrals.
- Key result: $Z_{\mathrm{TV},\mathcal{A}} = Z_{\mathrm{RT},\mathrm{Z}(\mathcal{A})}$, where $\mathrm{Z}(\mathcal{A})$ is the Drinfeld center of $\mathcal{A}$ (Reshetikhin-Virelizier (2010), Balsam-Kirillov (2010)).
- String nets: explicit construction of $Z_{\mathrm{TV},\mathcal{A}}(\Sigma)$ as ground-state of a local lattice Hamiltonian (Levin-Wen (2005)). Generalization of Kitaev's lattice gauge theory model of anyons (1997-2003).

Outline

1) Kitaev's lattice gauge model as a string net: *magnetic* picture
2) String nets from a fusion category
3) Kitaev's lattice gauge model as a string net: *electric* picture
4) Boundary excitations: the center construction

2D topological lattice gauge theories (Kitaev (2003))

Consider a planar graph, and a finite group G. The Hilbert space of the model is $\mathcal{H} = \mathcal{H}_{\mathrm{ZFC}}/\mathcal{N}$. $\mathcal{H}_{\mathrm{ZFC}}$ has an orthonormal basis of vectors $|\{g_{ij}\}\rangle$, with ij a link on the lattice, $g_{ij} = g_{ji}^{-1} \in G$, satisfying the zero flux condition: $g_{i_1 i_2}\, g_{i_2 i_3} \ldots g_{i_l i_1} = e$ for any plaquette bounded by l links.

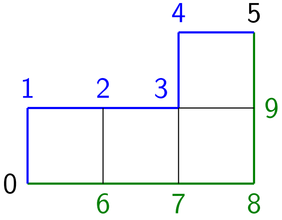
Gauge transformations: Pick $h_i \in G$ for each site i. Define $(\mathcal{T}_h\, g)_{ij} = h_i\, g_{ij}\, h_j^{-1}$. This transformation preserves the zero flux condition on all plaquettes. $\mathcal{N}$ is the subspace of $\mathcal{H}_{\mathrm{ZFC}}$ generated by vectors $|\{g_{ij}\}\rangle - |\{(\mathcal{T}_h\, g)_{ij}\}\rangle$.

$$\mathcal{H}_{\mathrm{ZFC}} = \mathcal{H}_{\mathrm{ZFC,S}} \oplus \mathcal{N}$$

So $\mathcal{H} = \mathcal{H}_{\mathrm{ZFC}}/\mathcal{N} \cong \mathcal{H}_{\mathrm{ZFC,S}} =$ ground-state of $(\mathrm{id} - \mathcal{P}_{\mathrm{S}})$.

2D topological lattice gauge theories (II)

Key fact: On a simply connected planar graph, any fluxless gauge configuration is related to the trivial one ($g_{ij} = e$) by a gauge transformation.



We wish to find $\{h_i\}$ such that $h_i\, g_{ij}\, h_j^{-1} = e$

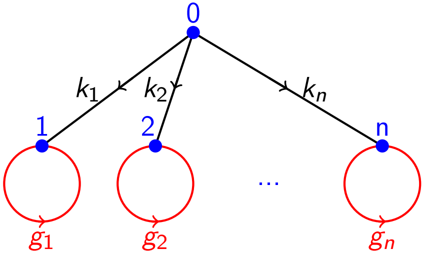
$$h_5 = h_0\, g_{01}\, g_{12}\, g_{23}\, g_{34}\, g_{45}$$

$$h_5 = h_0\, g_{06}\, g_{67}\, g_{78}\, g_{89}\, g_{95}$$

For a fluxless configuration, both paths give the same h_5: a non-Abelian and discrete version of Stokes' theorem. *Cohomological* viewpoint on 2D topological theories.

S^2 is simply connected, so $\mathcal{H}(S^2) = \mathbb{C}$. There exists a topological ground-state degeneracy on positive genus closed compact surfaces Σ, i.e. $\dim \mathcal{H}(\Sigma) \geq 2 \longrightarrow$ idea of topological quantum computation (Kitaev (1997-2003)).

Sphere with n holes



fluxless condition through complement of the holes:
$k_1 \, g_1 \, k_1^{-1} \cdots k_n \, g_n \, k_n^{-1} = e$
Gauge transformations:

$$k_i \rightarrow h_0 \, k_i \, h_i^{-1}$$
$$g_i \rightarrow h_i \, g_i \, h_i^{-1}$$

Setting $h_i = h_0 k_i$, we get $k_i = e$. So $\mathcal{H}(S^2, n) = \mathcal{H}_{\mathrm{ZFC}} / \mathcal{N}$, where $\mathcal{H}_{\mathrm{ZFC}}$ is spanned by basis vectors $|g_1, \cdots, g_n\rangle$ such that $g_1 \, g_2 \cdots g_n = e$, and $\mathcal{N}$ is generated by nul vectors $|g_1, \cdots, g_n\rangle - \left| h \, g_1 \, h^{-1}, \cdots, h \, g_n \, h^{-1} \right\rangle$ associated to gauge transformations.
If $n = 1$, $\dim(\mathcal{H}(S^2, 1))$ is equal to the number of conjugacy classes of G. For $n \geq 2$, we can fix conjugacy classes $\mathrm{Cl}_1, \mathrm{Cl}_2, ..., \mathrm{Cl}_n$ attached to the holes.

$$\mathcal{H}(S^2, n, \mathrm{Cl}_1, ..., \mathrm{Cl}_n) = \mathrm{Hom}_{Z(\mathrm{Vec}_G)}(1, (X(\mathrm{Cl}_1), \mathrm{id}) \otimes \cdots \otimes (X(\mathrm{Cl}_n), \mathrm{id}))$$

2D topological lattice gauge theories (III)

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Anyons from nonsolvable finite groups are sufficient for universal quantum computation

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(Received 1 October 2002; published 28 February 2003)

We present a constructive proof that anyonic magnetic charges with fluxes in a nonsolvable finite group can perform universal quantum computations. The gates are built out of the elementary operations of braiding, fusion, and vacuum pair creation, supplemented by a reservoir of ancillas of known flux. Procedures for building the ancilla reservoir and for correcting leakage are also described. Finally, a universal qudit gate set, which is ideally suited for anyons, is presented. The gate set consists of classical computation supplemented by measurements of the X operator.

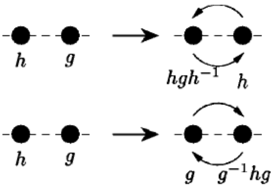


FIG. 1. Exchanging two anyons.

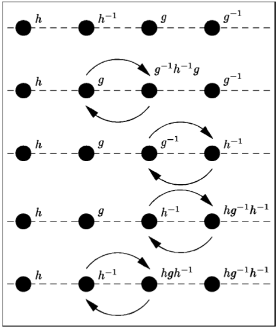
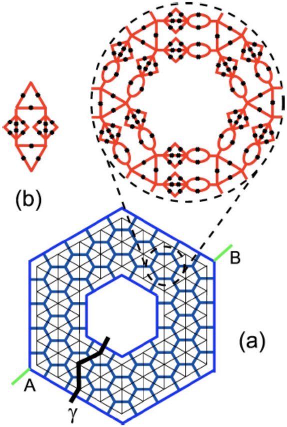


FIG. 2. Conjugating a pair of anyons.

Proposed implementation with Josephson circuits



G is the permutation group $\mathcal{S}_3$

Douçot, Ioffe, Vidal, PRB 69, 214501 (2005)

Outline

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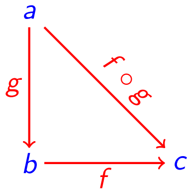
Basics of categories

"*Since a category consists of arrows, our subject could also be described as learning how to live without elements, using arrows instead.*" S. Mac Lane, Categories for the working mathematician (1971)



$f \in \mathrm{Hom}(\mathrm{b}, \mathrm{c})$, $g \in \mathrm{Hom}(\mathrm{a}, \mathrm{b})$
$f \circ g \in \mathrm{Hom}(\mathrm{a}, \mathrm{c})$
$\mathrm{id}_a \in \mathrm{Hom}(a, a)$, $\mathrm{id}_b \in \mathrm{Hom}(b, b)$
$g \circ \mathrm{id}_a = g = \mathrm{id}_b \circ g$

Example: Consider an ordered set $(S, \leq)$. It defines a category $\mathcal{C}$, whose objects are elements of S and $\mathrm{Hom}(a, b)$ contains a unique arrow if $a \leq b$, and is empty otherwise.

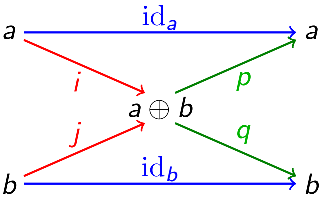
$\mathbb{C}$-linear categories

- $\mathrm{Hom}(a, b)$ is a finite dimensional vector space over $\mathbb{C}$, such that composition of arrows is $\mathbb{C}$-bilinear:

$$h\circ(\lambda f+\mu g)=\lambda(h\circ f)+\mu(h\circ g)$$
$$(\lambda h+\mu k)\circ f=\lambda(h\circ f)+\mu(k\circ g)$$

$f, g \in \mathrm{Hom}(a, b)$
$h, k \in \mathrm{Hom}(b, c)$
$\lambda, \mu \in \mathbb{C}$.

- Existence of a zero object 0, such that $\mathrm{Hom}(0, 0) = 0 = \{\mathrm{id}_0\}$.
- Existence of direct sums $a \oplus b$.



$p\circ i=\mathrm{id}_a$, $q\circ j=\mathrm{id}_b$
$q\circ i=0$, $p\circ j=0$
$i\circ p+j\circ q=\mathrm{id}_{a\oplus b}$

$\mathbb{C}$-linear categories (II)

Important consequence:

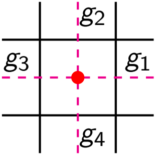
$$\begin{aligned} \mathrm{Hom}\Big(\bigoplus_\alpha a_\alpha, \bigoplus_\beta b_\beta\Big) &\cong \bigoplus_{\alpha,\beta} \mathrm{Hom}(a_\alpha, b_\beta) \\ f &\mapsto \{f_{\alpha,\beta} = p_\beta \circ f \circ i_\alpha\} \end{aligned}$$

Fusion categories

- Each object X is a finite direct sum of simple objects X_i: $X = \bigoplus_i n_i X_i$. $\mathrm{Hom}(X_i, X_j) = 0$ if $i \neq j$ and ($\mathbb{C}$ alg. closed) $\mathrm{Hom}(X_i, X_i) = \mathbb{C}\,\mathrm{id}_{X_i}$.
- There are finitely many simple objects (modulo isomorphisms).

First contact with string net models

Plaquette of lattice gauge model → site on the dual lattice. Zero flux condition $g_1\, g_2\, g_3\, g_4 = e$ at each dual lattice site.



The Vec_G category

- Objects: G-graded vector spaces $V = \bigoplus_{g\in G} V_g$ over $\mathbb{C}$
- Arrows from V to W: Collection of linear maps $f_g : V_g \to W_g$
- Simple objects: δ_g such that $(\delta_g)_h = 0$ if $g \neq h$ and $(\delta_g)_g = \mathbb{C}$.

First step: assign an object of Vec_G to each link of (dual) lattice.
But: how to implement the zero flux condition at (dual) lattice sites?

$g_1\, g_2\, g_3\, g_4 = e \Leftrightarrow \mathrm{Hom}(\mathbb{C}, \delta_{g_1} \otimes \delta_{g_2} \otimes \delta_{g_3} \otimes \delta_{g_4}) \neq 0$

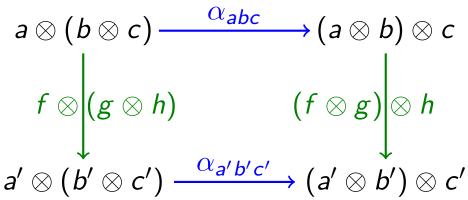
Each site satisfies the Vec_G fusion rules.

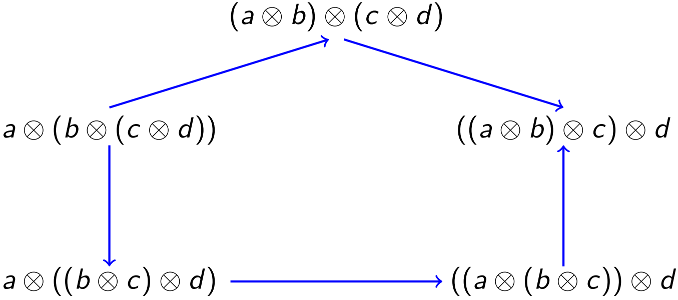
Further requests for $\mathcal{A}$ (input category) from string nets

Fusion rule for $\mathcal{A} = \mathrm{Vec}_G$ involves $\mathrm{Hom}(\mathbb{C}, \delta_{g_1} \otimes \delta_{g_2} \otimes \delta_{g_3} \otimes \delta_{g_4})$. For a general $\mathcal{A}$:

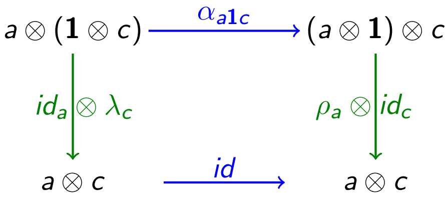
- $\mathbb{C}$ is replaced by a unit object, denoted by $\mathbf{1}$.
- We need a notion of tensor product.
  In Vec_G: $(V \otimes W)_g = \bigoplus_h (V_h \otimes W_{h^{-1}g})$.
- A link is adjacent to two sites. We need to swap orientation:
  $g_{ij} \to g_{ji} = g_{ij}^{-1}$ becomes $V_{ij} \to V_{ij}^*$: notion of duality.

Tensor products: associativity constraints





Tensor products: the unit object



Mac Lane Coherence theorem: Consider words composed of objects in $\mathcal{A}$, tensor product signs, and parentheses. Pick a pair of words, involving the same sequences of objects, but differing in terms of location of parentheses and of possible of ocurrences of $\mathbf{1}$. Example: $(((a \otimes \mathbf{1}) \otimes (b \otimes c)) \otimes \mathbf{1}) \otimes d$ and $a \otimes (b \otimes (c \otimes d))$. It is possible to connect them by several different sequences of arrows, involving α, λ and ρ isomorphisms. Then: all such sequences induce the same arrow between these two words.

Duality (I)

V^* is a left dual for V if we have two arrows $\mathrm{ev}_V : V^* \otimes V \to \mathbf{1}$ and $\mathrm{coev}_V : \mathbf{1} \to V \otimes V^*$ such that (rigidity):

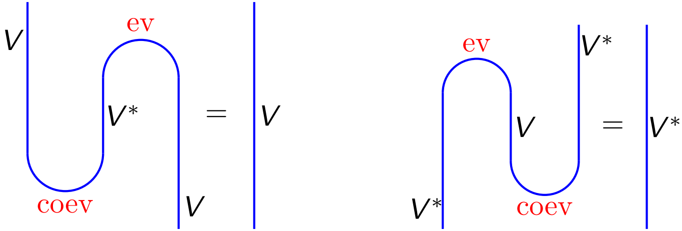
$$V \xrightarrow{\lambda_V^{-1}} \mathbf{1} \otimes V \xrightarrow{\mathrm{coev} \otimes \mathrm{id}} (V \otimes V^*) \otimes V \xrightarrow{\alpha^{-1}} V \otimes (V^* \otimes V) \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} V \otimes \mathbf{1} \xrightarrow{\rho_V} V = \mathrm{id}_V$$

$$V \xrightarrow{\rho_{V^*}^{-1}} V^* \otimes \mathbf{1} \xrightarrow{\mathrm{id} \otimes \mathrm{coev}} V^* \otimes (V \otimes V^*) \xrightarrow{\alpha} (V^* \otimes V) \otimes V^* \xrightarrow{\mathrm{ev} \otimes \mathrm{id}} \mathbf{1} \otimes V^* \xrightarrow{\lambda_{V^*}} V^* = \mathrm{id}_{V^*}.$$

Vec: (finite dimensional vector spaces). $\mathrm{ev}_V : V^* \otimes V \to \mathbb{C}$ sends $\varphi \otimes v$ into $\varphi(v)$. Pick dual bases $\{\alpha_i\}$, $\{e_j\}$ for V^* and V, i.e. $\alpha_i(e_j) = \delta_{ij}$. $\mathrm{coev}_V : \mathbb{C} \to V \otimes V^*$ sends $1 \in \mathbb{C}$ into $\sum_i e_i \otimes \alpha_i$.
Rigidity: $v = \sum_i \alpha_i(v)\, e_i$ and $\varphi = \sum_i \varphi(e_i)\, \alpha_i$ for any $v \in V$ and $\varphi \in V^*$.

Vec_G: $\mathbf{1} = \delta_e$. $(V^*)_g = (V_{g^{-1}})^*$.

Graphical representation of duality axiom



Duality (II)

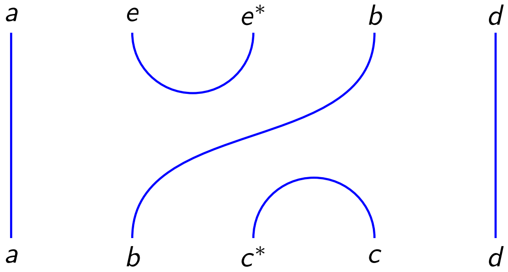
Extended coherence theorem: Consider words composed of objects in $\mathcal{A}$, tensor product signs, and parentheses. Pick a pair of words, differing in terms of location of parentheses and of possible of ocurrences of $\mathbf{1}$, but also via possible annihilation (resp. creation) of a^*a (resp aa^*) pairs. It is possible to connect them by several different sequences of arrows, involving α, λ, ρ isomorphisms, and ev and coev arrows. Then: all such sequences induce the same arrow between these two words.

Example:

$((a \otimes (b \otimes \mathbf{1})) \otimes c^*) \otimes ((c \otimes d) \otimes \mathbf{1}) \to (a \otimes e) \otimes ((e^* \otimes b) \otimes d)$

Graphical representation

$$((a \otimes (b \otimes \mathbf{1})) \otimes c^*) \otimes ((c \otimes d) \otimes \mathbf{1}) \to (a \otimes e) \otimes ((e^* \otimes b) \otimes d)$$



Expression of arrows from simple objects

$A = \bigoplus_i n_i X_i$, where $n_i = \dim \mathrm{Hom}(A, X_i) = \dim \mathrm{Hom}(X_i, A)$. Consider dual bases $\{u_{i\alpha}\}$ for $\mathrm{Hom}(X_i, A)$, and $\{v_{i\alpha}\}$ for $\mathrm{Hom}(A, X_i)$, i.e. $v_{i\alpha} \circ u_{i\alpha} = \delta_{ij}\,\mathrm{id}_{X_i}$. Then:

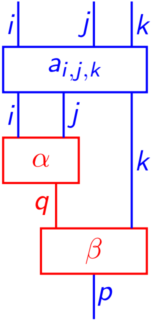
$$\mathrm{id}_A = \sum_{i,\alpha} u_{i\alpha} \circ v_{i\alpha}$$

Consider $f : A \longrightarrow B$. $f = f \circ \mathrm{id}_A = \sum_{i,\alpha} (f \circ u_{i\alpha}^{(A)}) \circ v_{i\alpha}^{(A)}$.
$f \circ u_{i\alpha}^{(A)} = \sum_\mu u_{i\mu}^{(B)} \circ (v_{i\mu}^{(B)} \circ f \circ u_{i\alpha}^{(A)})$.
$v_{i\mu}^{(B)} \circ f \circ u_{i\alpha}^{(A)} = \langle \mu | F_i^f | \alpha \rangle\, \mathrm{id}_{X_i}$

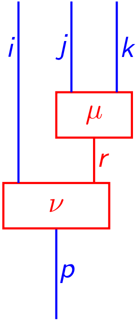
$$f \circ u_{i\alpha}^{(A)} = \sum_\mu \left\langle \mu | F_i^f | \alpha \right\rangle u_{i\mu}^{(B)}$$

F symbols for associativity isomorphisms

$A = (X_i \otimes X_j) \otimes X_k,\ B = X_i \otimes (X_j \otimes X_k)$ and $f = a_{X_i, X_j, X_k}$.



$$= \sum_{r,\mu,\nu} \left\langle r, \mu, \nu \middle| F_p^{ijk} \middle| q, \alpha, \beta \right\rangle$$

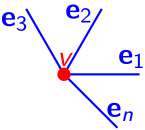


Definition of $\mathcal{H}_{\mathrm{FR}}$

Generalization of $\mathcal{H}_{\mathrm{ZFC}}$, defined for $\mathcal{A} = \mathrm{Vec}_G$. Inspired directly by A. Kirillov, Jr., arXiv:1106.6033.

States $|\{g_{ij}\}\rangle$ are replaced by $|\{V(\mathbf{e}), \varphi(v)\}\rangle$.

- For each edge $\mathbf{e}$ choose an object $V(\mathbf{e})$ in $\mathcal{A}$.
- Arrow reversal: $V(\bar{\mathrm{e}}) = V(\mathbf{e})^*$
- For each vertex v choose $\varphi(v) \in \mathrm{Hom}(\mathbf{1}, V(\mathbf{e}_1) \otimes ... \otimes V(\mathbf{e}_n))$.

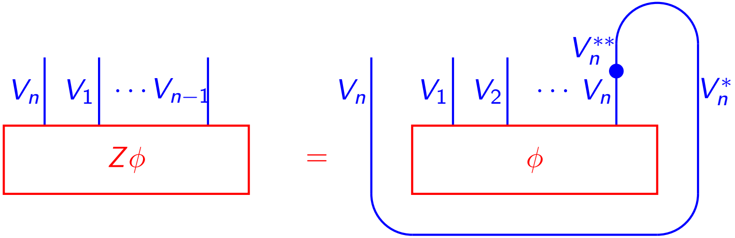


Notion of isomorphism between $\{V(\mathbf{e}), \varphi(v)\}$ and $\{V'(\mathbf{e}), \varphi'(v)\}$: Defined by a collection of isomorphisms $f_{\mathbf{e}_j} : V(\mathbf{e}_j) \to V'(\mathbf{e}_j)$, such that: $\varphi'(v) = (f_{\mathbf{e}_1} \otimes ... \otimes f_{\mathbf{e}_n}) \circ \varphi(v)$.

Cyclic permutation symmetry around a vertex (I)

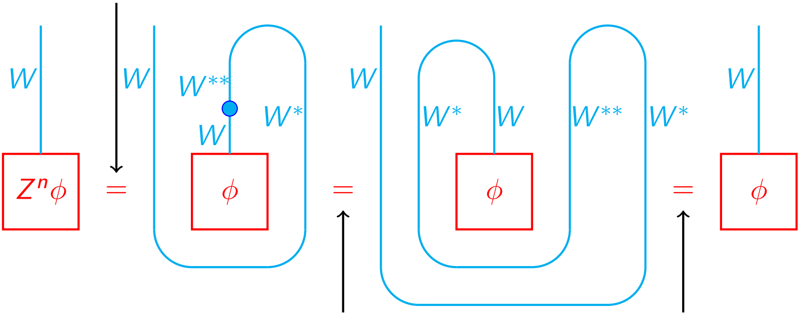
$$\begin{array}{ccc}
\mathrm{Hom}(\mathbf{1}, V_1 \otimes \ldots \otimes V_{n-1} \otimes V_n) & \xrightarrow{Z} & \mathrm{Hom}(\mathbf{1}, V_n \otimes V_1 \otimes \ldots \otimes V_{n-1}) \\
\downarrow (f_1 \otimes \ldots \otimes f_{n-1} \otimes f_n) \circ . & & \downarrow (f_n \otimes f_1 \otimes \ldots \otimes f_{n-1}) \circ . \\
\mathrm{Hom}(\mathbf{1}, V_1' \otimes \ldots \otimes V_{n-1}' \otimes V_n') & \xrightarrow{Z'} & \mathrm{Hom}(\mathbf{1}, V_n' \otimes V_1' \otimes \ldots \otimes V_{n-1}')
\end{array}$$

Pivotal structure $\delta_V : V \to V^{**}$



Cyclic permutation symmetry around a vertex: $Z^n = \mathrm{id}$

Consequence of $\delta_{V_1 \otimes V_2} = \delta_{V_1} \otimes \delta_{V_2}$ (set $W = V_1 \otimes \cdots \otimes V_n$)

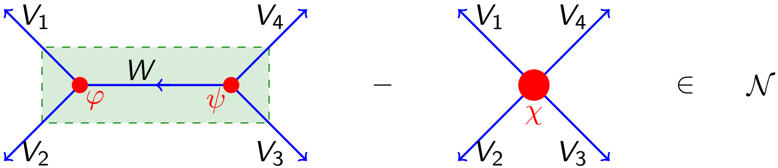


δ is a natural transformation

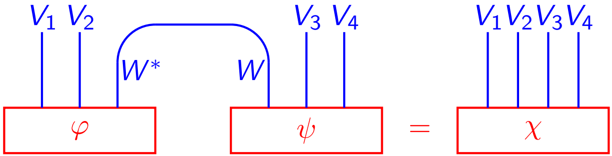
rigidity

Definition of the $\mathcal{N}$ subspace (I)

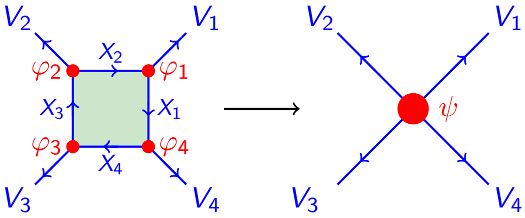
Goal: define local updates of $\{V(\mathbf{e}), \varphi(v)\}$, which *do not change* the state of the system outside of a finite connected region.



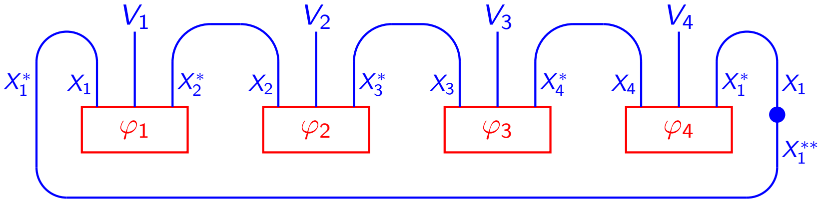
How to assign a meaning to this notion?



Definition of the $\mathcal{N}$ subspace (II)

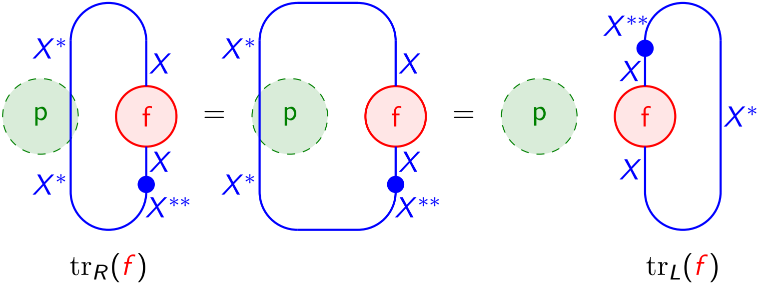


where ψ is given by:



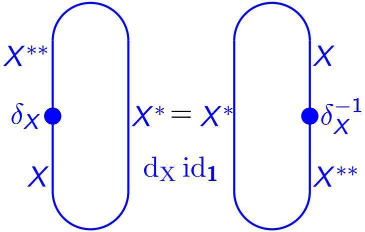
$\mathcal{H}$ on a sphere

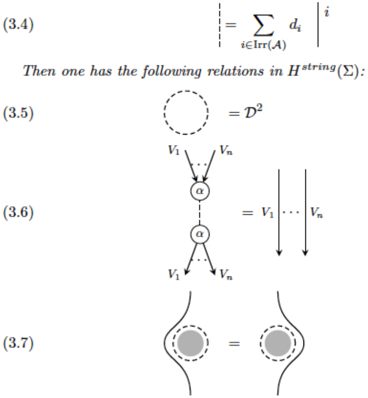
$\mathcal{H}(S^2 - \{p\}) = \mathcal{H}(\mathbb{R}^2) = \mathrm{Hom}_{\mathcal{A}}(\mathbf{1}, \mathbf{1}) = \mathbb{C}$
$\pi : \mathcal{H}(S^2 - \{p\}) \longrightarrow \mathcal{H}(S^2)$ surjective so $\dim \mathcal{H}(S^2) \leq 1$.
When do we have $\dim \mathcal{H}(S^2) = 1$?
A: Constraint near p should always be satisfied.



Equality holds when $\mathcal{A}$ is spherical, i.e. when $\mathrm{tr}_L(f) = \mathrm{tr}_L(f)$ for any arrow f.

Dimension of objects

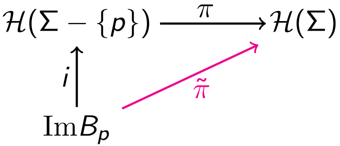




Kirillov (2011)

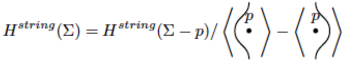
The Levin-Wen projector (I)

FIGURE 6. Operator B_p

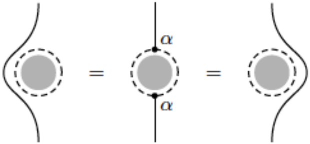


$B_p\psi - \psi \in \mathcal{N}(\Sigma)$ for any $\psi \in \mathcal{H}(\Sigma - \{p\})$, so $\tilde{\pi}$ is surjective.

Description of $\mathrm{Ker}\,\pi$

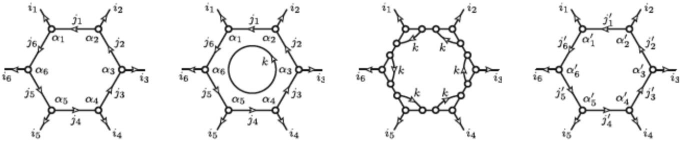


Kirillov (2011)



If $\psi \in \mathrm{Ker}\,\pi$ then $B_p\psi = 0$, so $\tilde{\pi}$ is injective.

The Levin-Wen projector (II)



(a) **(b)** **(c)** **(d)**

Fig. 3. The action of the plaquette operator $B_{\mathbf{p}}^k$: a) the initial state of the plaquette; b) a symbolic representation of the operator $B_{\mathbf{p}}^k$ applied to it; c) the loop is partially fused using Eq. (12) (some labels and the overall factor are not shown); d) the corner triangles have been evaluated to trivalent vertices (summation over j'_p, α'_q is assumed)

A. Kitaev and Liang Kong, Comm. Math. Phys. 313, 351 (2012)

Outline

1) Kitaev's lattice gauge model as a string net: *magnetic* picture
2) String nets from a fusion category
3) Kitaev's lattice gauge model as a string net: *electric* picture
4) Boundary excitations: the center construction

Motivation for Rep_G string-net

On a given link, associated Hilbert space is $\mathbb{C}[G] = \bigoplus_{g \in G} \mathbb{C} |g\rangle$.
Left action of G on $\mathbb{C}[G]$: $L_h |g\rangle = |hg\rangle$.
Right action of G on $\mathbb{C}[G]$: $R_h |g\rangle = |gh^{-1}\rangle$.
These two actions commute.

Gauge transformation: $\mathcal{T}(h_i, h_j) = L_{h_i} \circ R_{h_j}$

Gauge invariant constraint at site i: project on singlet subspace of $\bigotimes_j \mathbb{C}[G]_{ij}$ for the L_{h_i} action, equivalent to choose an element in $\mathrm{Hom}_{\mathrm{Rep}_G}(\mathbb{C}, \bigotimes_j \mathbb{C}[G]_{ij})$, i.e. to satisfy Rep_G fusion rule at site i.

Basics of Rep_G (I)

Objects: Finite dimensional representations (E, ρ) of G.
Arrows: $\mathrm{Hom}_{\mathrm{Rep}_G}((E, \rho), (F, \sigma))$ is composed of linear maps $f : E \longrightarrow F$ such that

$$\begin{array}{ccc} E & \xrightarrow{\quad \rho_g \quad} & E \\ f \downarrow & & \downarrow f \\ F & \xrightarrow{\quad \sigma_g \quad} & F \end{array}$$

commutes for any g in G.

Tensor product: $(E, \rho) \otimes (F, \sigma) \cong (E \otimes F, \rho \otimes \sigma)$.
Unit object: $\mathbf{1} = (\mathbb{C}, \mathrm{id})$.

$\mathrm{Hom}_{\mathrm{Rep}_G}(\mathbf{1}, (E, \rho)) \cong \{v \in E, \rho_g(v) = v, \forall g \in G\}$: Invariant subspace of E under ρ.

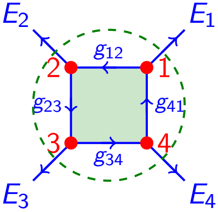
Basics of Rep_G (II)

Duality: $(E, \rho)^* = (E^*, \rho^*)$, where $\rho^*_g = \rho^{\mathrm{T}}_{g^{-1}}$.
Exercise: Check that $\mathrm{ev}_E : E^* \otimes E \to \mathbb{C}$ and $\mathrm{coev}_E : \mathbb{C} \to E \otimes E^*$ defined in Vec also define arrows in Rep_G.

Simple objects: Finite dimensional irreducible representations (E_i, ρ_i) of G.

Classical decomposition of $\mathbb{C}[G]$:
As a vector space: $\mathbb{C}[G] = \bigoplus_i E_i \otimes E_i^*$
Left action: $\bigoplus_i (E_i, \rho_i) \otimes (E_i^*, \mathrm{id})$
Right action: $\bigoplus_i (E_i, \mathrm{id}) \otimes (E_i^*, \rho_i^*)$
Used in Buerschaper and Aguado PRB 80, 155136 (2009).

Fluxless constraint in Rep_G (I)



$$\mathcal{H} = (\bigotimes_{i=1}^{n} E_i) \otimes \left(\bigoplus_{\{g_{ij}\}} \mathbb{C} \, |\{g_{ij}\}\rangle \right)$$

Rep_G string-net prescription:
First apply gauge invariance at vertices, to get $\mathcal{H}_{FR}$, using:
$\mathcal{T}(\{h_i\})(v \otimes |\{g_{ij}\}\rangle) = (\otimes_i \rho_{i,h_i})(v) \otimes \left| \{ h_i \, g_{ij} \, h_j^{-1} \} \right\rangle$
Then form $\mathcal{H}_{FR}/\mathcal{N} \cong \mathrm{Hom}_{\mathrm{Rep}_G}(\mathbf{1}, \bigotimes_i (E_i, \rho_i))$.

Question: Is this equivalent to imposing the fluxless constraint:
$g_{12} \, g_{23} \cdots g_{n-1,n} \, g_{n1} = e$?

Fluxless constraint in Rep_G (II)

$$\mathcal{H} = \left(\bigotimes_{i=1}^{n} E_i\right) \bigotimes \left(\bigoplus_{\{g_{ij}\}} \mathbb{C}\,|\{g_{ij}\}\rangle\right)$$

Fluxless constraint: $g_{12}\, g_{23} \ldots g_{n-1,n}\, g_{n1} = e$ defines $\mathcal{H}_{\mathrm{ZFC}}$.
Gauge action: $\mathcal{T}(\{h_i\})(v \otimes |\{g_{ij}\}\rangle) = (\otimes_i \rho_{i,h_i})(v) \otimes \left|\{h_i\, g_{ij}\, h_j^{-1}\}\right\rangle$

In fluxless sector, we can bring $\{g_{ij}\}$ to the trivial configuration $\{g_{ij} = e\}$ by a gauge transformation, which has for stabilizor $\{h_i = h\}$, i.e. the diagonal subgroup in $G_1 \times \ldots \times G_n$.

Invariant states in $\mathcal{H}_{\mathrm{ZFC}}$ are in 1 to 1 correspondence with invariant states in $\bigotimes_i E_i$ under $\rho = \bigotimes_i \rho_i$, that is $\mathrm{Hom}_{\mathrm{Rep}_G}(\mathbf{1}, \bigotimes_i (E_i, \rho_i))$.

This is the expected image subspace of the plaquette projector in the Rep_G string-net model.

Morita equivalence

Models for Gapped Boundaries and Domain Walls

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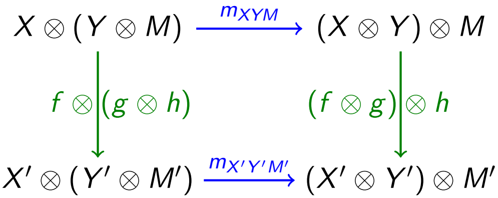
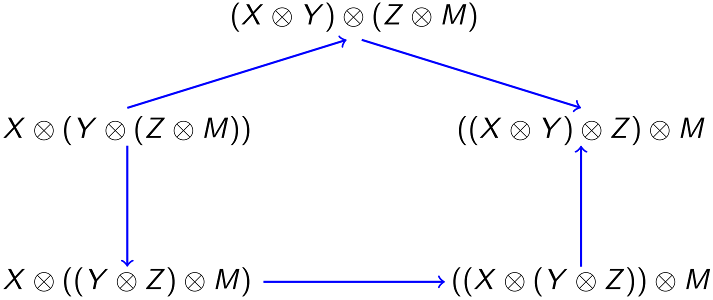
Abstract: We define a class of lattice models for two-dimensional topological phases with boundary such that both the bulk and the boundary excitations are gapped. The bulk part is constructed using a unitary tensor category $\mathcal{C}$ as in the Levin-Wen model, whereas the boundary is associated with a module category over $\mathcal{C}$. We also consider domain walls (or defect lines) between different bulk phases. A domain wall is transparent to bulk excitations if the corresponding unitary tensor categories are Morita equivalent. Defects of higher codimension will also be studied. In summary, we give a dictionary between physical ingredients of lattice models and tensor-categorical notions.

Comm. Math. Phys. 313, 351 (2012)

Module categories: associativity constraints



X, Y, Z, X', Y' in $\mathcal{C}$
M, M' in $\mathcal{M}$



Module categories: behavior of the unit object

$$\begin{array}{ccc}
X \otimes (\mathbf{1} \otimes M) & \xrightarrow{\alpha_{X\mathbf{1}M}} & (a \otimes \mathbf{1}) \otimes c \\
\Big\downarrow {\scriptstyle id_X \otimes \lambda_M} & & \Big\downarrow {\scriptstyle \rho_X \otimes id_M} \\
X \otimes M & \xrightarrow{\quad id \quad} & X \otimes M
\end{array}$$

Important example of module category:

$\mathcal{M} = \mathrm{Vec}$ is a module category over $\mathcal{C} = \mathrm{Vec}_G$ and also over $\mathcal{D} = \mathrm{Rep}_G$.

Module categories and line defects

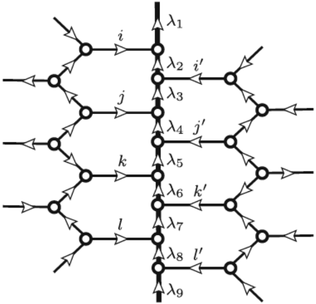


Fig. 8. A neighborhood of a defect line between two topological phases, where $i, j, k, l \in \mathcal{C}$, $\lambda_1, \ldots, \lambda_9 \in \mathcal{M}$, $i', j', k', l' \in \mathcal{D}$.

Kitaev and Kong (2012)

Outline

1) Kitaev's lattice gauge model as a string net: *magnetic* picture
2) String nets from a fusion category
3) Kitaev's lattice gauge model as a string net: *electric* picture
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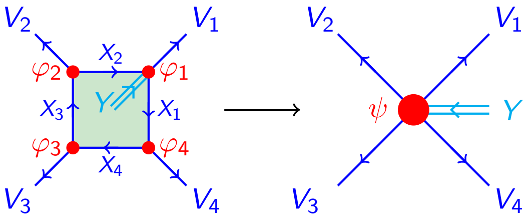
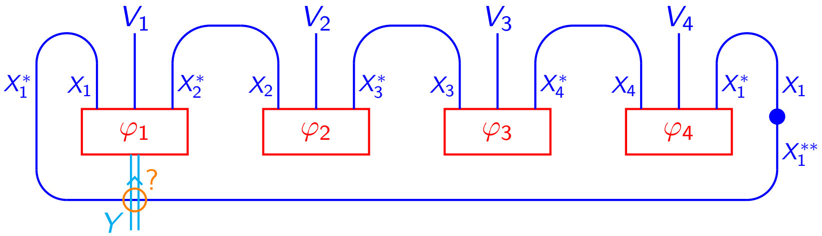
Point excitations in string-net models

Flux: Vec_G fusion rule violated at a dual lattice site.

Charge: Rep_G fusion rule violated at an original lattice site.





Definition of $Z(\mathcal{A})(I)$

Objects of $Z(\mathcal{A})$ are pairs (X, σ) with X object in $\mathcal{A}$ and σ is an half-braiding, i.e. a collection of arrows $\sigma_V : V \otimes X \longrightarrow X \otimes V$ defined for any object V in $\mathcal{A}$, subject to two conditions:

Naturality:

$$\begin{array}{ccc}
V \otimes X & \xrightarrow{\sigma_V} & X \otimes V \\
{\scriptstyle f \otimes \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_X \otimes f} \\
W \otimes X & \xrightarrow{\sigma_W} & X \otimes W
\end{array}$$

commutes for all $f \in \mathrm{Hom}_{\mathcal{A}}(V, W)$.

Compatibility with tensor product: $\sigma_{V \otimes W}$ is given by:

$$\begin{array}{ccccc}
(V \otimes W) \otimes X & \xrightarrow{\alpha_{V,W,X}} & V \otimes (W \otimes X) & \xrightarrow{\mathrm{id}_V \otimes \sigma_W} & V \otimes (X \otimes W) \\
{\scriptstyle \sigma_{V \otimes W}}\downarrow & & & & \downarrow{\scriptstyle \alpha^{-1}_{V,X,W}} \\
X \otimes (V \otimes W) & \xleftarrow{\alpha_{X,V,W}} & (X \otimes V) \otimes W & \xleftarrow{\sigma_V \otimes \mathrm{id}_W} & (V \otimes X) \otimes W
\end{array}$$

Definition of $Z(\mathcal{A})(II)$

An arrow f of $Z(\mathcal{A})$ from (X, σ) to (Y, τ) is an arrow $f \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$ such that:

$$\begin{array}{ccc} V \otimes X & \xrightarrow{\sigma_V} & X \otimes V \\ \Big\downarrow \mathrm{id}_V \otimes f & & \Big\downarrow f \otimes \mathrm{id}_V \\ V \otimes Y & \xrightarrow{\tau_V} & Y \otimes V \end{array}$$

commutes for any object V in $\mathcal{A}$.

$Z(\mathcal{A})$ has tensor products

$(X,\sigma)\otimes(Y,\tau) = (X\otimes Y, \tau.\sigma)$, where $\tau.\sigma$ is the half-braiding defined by:

$$\begin{array}{ccccc} V\otimes(X\otimes Y) & \xrightarrow{\alpha^{-1}_{V,X,Y}} & (V\otimes X)\otimes Y & \xrightarrow{\sigma_V\otimes \mathrm{id}_Y} & (X\otimes V)\otimes Y \\ \Big\downarrow{\scriptstyle (\tau.\sigma)_V} & & & & \Big\downarrow{\scriptstyle \alpha_{X,V,W}} \\ (X\otimes Y)\otimes V & \xleftarrow{\alpha^{-1}_{X,Y,V}} & X\otimes(Y\otimes V) & \xleftarrow{\mathrm{id}_X\otimes\tau_V} & X\otimes(V\otimes Y)\end{array}$$

Exercises:

- Check that $\tau.\sigma$ satisfies the two constraints (naturality and compatibility with tensor product) involved in the definition of a half-braiding.
- For $f : (X,\sigma)\to(X',\sigma')$, $g :\to (Y,\tau),(Y',\tau')$, check that $f\otimes g : (X,\sigma)\otimes(Y,\tau)\to(X',\sigma')\otimes(Y',\tau')$ also defines an arrow in $Z(\mathcal{A})$.

Associativity of tensor product and unit object in $Z(\mathcal{A})$

$$\begin{array}{ccc} ((X,\rho)\otimes(Y,\sigma))\otimes(Z,\tau) & & (X,\rho)\otimes((Y,\sigma)\otimes(Z,\tau)) \\ \wr\| & & \wr\| \\ ((X\otimes Y)\otimes Z, \tau.(\sigma.\rho)) & \xrightarrow{\alpha_{X,Y,Z}} & (X\otimes(Y\otimes Z), (\tau.\sigma).\rho) \end{array}$$

Exercises:

- Check that $\alpha_{X,Y,Z}$ defines an arrow from $((X\otimes Y)\otimes Z, \tau.(\sigma.\rho))$ to $(X\otimes(Y\otimes Z), (\tau.\sigma).\rho)$ in $Z(\mathcal{A})$.
- Check that naturality of α and pentagon identify transfer from $\mathcal{A}$ to $Z(\mathcal{A})$.
- Check that $(1,\tau)$ with $\tau_V = l_V^{-1}\circ r_V : V\otimes 1 \longrightarrow V \longrightarrow 1\otimes V$ is an object in $Z(\mathcal{A})$.
- Check that $(1,\tau)$ is a unit object with respect to the tensor product in $Z(\mathcal{A})$.

Description of $Z(\mathrm{Vec}_G)$ (I)

Naturality: (X, σ) is determined by a collection of linear maps $\sigma_{g,k} : X_k \cong (\delta_g \otimes X)_{gk} \longrightarrow (X \otimes \delta_g)_{gk} \cong X_{gkg^{-1}}$
Minimal objects of $Z(\mathrm{Vec}_G)$ are supported on given conjugacy class $\mathrm{Cl}(k)$ of G.

Compatibility with tensor product: $\sigma_{gh,k} = \sigma_{g,hkh^{-1}} \circ \sigma_{g,k}$
If $g, h \in \mathrm{Stab}(k)$, $\sigma_{gh,k} = \sigma_{g,k} \circ \sigma_{g,k}$, so we get a representation ρ of $\mathrm{Stab}(k)$, acting on a $\mathbb{C}$ vector space E.

Description of (X, σ): Pick a set of representatives $\{g_i\}$ so that any element in $\mathrm{Cl}(k)$ may be uniquely written as $g_i k g_i^{-1}$. Then:

$$\begin{aligned} X_{g_i k g_i^{-1}} &= \mathbb{C}e_i \otimes E \\ \sigma_h(e_i \otimes v) &= e_j \otimes \rho(s)(v) \\ hg_i &= g_j s, s \in \mathrm{Stab}(k) \end{aligned}$$

Description of $Z(\mathrm{Vec}_G)$ (II)

Arrows $\hat{f}$ from (X, σ) to (Y, τ):

- If (X, σ) and (Y, τ) are supported on different conjugacy classes: $\mathrm{Hom}_{Z(\mathrm{Vec}_G)}((X, \sigma), (Y, \tau)) = 0$.
- If (X, σ) and (Y, τ) are both supported on $\mathrm{Cl}(k)$: $\mathrm{Hom}_{Z(\mathrm{Vec}_G)}((X, \sigma), (Y, \tau)) = \mathrm{Hom}_{\mathrm{Rep}(\mathrm{Stab}(k))}(\rho_\sigma, \rho_\tau)$.

$$\begin{array}{ccc} E & \xrightarrow{\rho_\sigma(s)} & E \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ F & \xrightarrow{\rho_\tau(s)} & F \end{array} \qquad \text{commutes for all } s \in \mathrm{Stab}(k)$$

$$\hat{f}(e_i \otimes v) = e_j \otimes f(v).$$

Magnetic flux excitations in $Z(\mathrm{Vec}_G)$

Magnetic flux excitations correspond to choosing the identity representation of $\mathrm{Stab}(k)$: $E = \mathbb{C}$ and $\mathrm{id}(s) = \mathrm{id}_{\mathbb{C}}$ for all $s \in \mathrm{Stab}(k)$. The corresponding object $X(\mathrm{Cl}(k)) = \bigoplus_{g \in \mathrm{Cl}(k)} \mathbb{C} \, |g\rangle$. Then:

$$\sigma_h(|g\rangle) = \left|hgh^{-1}\right\rangle$$

Tensor product of magnetic flux excitations:
$(X(\mathrm{Cl}_1), \mathrm{id}) \otimes \cdots \otimes (X(\mathrm{Cl}_n), \mathrm{id})$ is associated to the G-graded vector space $X = \bigoplus_{g_i \in \mathrm{C}_i} \mathbb{C} \, |g_1, ..., g_n\rangle$. The grading is defined by $|g_1, ..., g_n\rangle \in X_{g_1 \dots g_n}$.

$$\sigma_h \, |g_1, ..., g_n\rangle = \left|hg_1h^{-1}, ..., hg_nh^{-1}\right\rangle$$

$\mathrm{Hom}_{Z(\mathrm{Vec}_G)}(1, (X(\mathrm{Cl}_1), \mathrm{id}) \otimes \cdots \otimes (X(\mathrm{Cl}_n), \mathrm{id}))$

Motivation: Space of states on a sphere with n punctures, carrying magnetic flux excitations associated to $\mathrm{Cl}_1, \cdots, \mathrm{Cl}_n$ conjugacy classes.

Define $(X, \sigma) = (X(\mathrm{Cl}_1), \mathrm{id}) \otimes \cdots \otimes (X(\mathrm{Cl}_n), \mathrm{id})$.

$$\begin{aligned}
\mathrm{Hom}_{Z(\mathrm{Vec}_G)}(1, (X, \sigma)) &= \mathrm{Hom}_{Z(\mathrm{Vec}_G)}(1, (X_e, \sigma_e)) \\
&= \mathrm{Hom}_{\mathrm{Rep(G)}}(\mathrm{id}, \sigma_e) \\
&= \{v \in X_e | \forall h \in G, \sigma_h(v) = v\}
\end{aligned}$$

- Basis for X_e: $\{|g_1, ..., g_n\rangle \, | g_i \in \mathrm{Cl}_i, g_1 ... g_n = e\}$.
- σ_h permutes basis vectors.
- Dimension of invariant vectors subspace = number of orbits of basis vectors under σ_h permutations (gauge transformations) = original lattice gauge theory count.

Equivalence between $Z(\mathcal{A})$ and $\mathrm{Rep}(TA)$

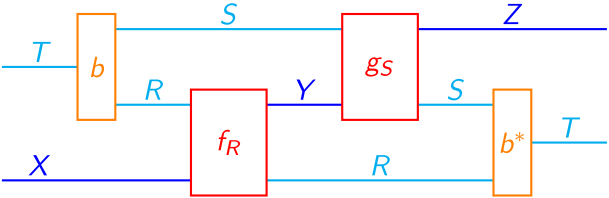
Developed in Lan and Wen (PRB (2014)).
General proof for $\mathcal{A}$ spherical fusion category given by Popa, Shlyakhtenko, Vaes (2018).

Useful, because $\mathrm{Rep}(TA)$ is semi-simple, i.e. any representation of $T(\mathcal{A})$ can be decomposed as a direct sum of irreducible representations (Müger (2003)).

I will follow the presentation of Hardiman (arXiv:1911.07271). He introduces a category $\mathcal{T}(\mathcal{A})$ called the tube category of $\mathcal{A}$, and a related category $\mathcal{RT}(\mathcal{A})$. He shows separately equivalence between $Z(\mathcal{A})$ and $\mathcal{RT}(\mathcal{A})$ and then between $\mathcal{RT}(\mathcal{A})$ and $\mathrm{Rep}(TA)$.

The tube category $\mathcal{T}(\mathcal{A})$

- Objects are the same as the objects of $\mathcal{A}$.
- Arrows are different:
  $\mathrm{Hom}_{\mathcal{T}(\mathcal{A})}(X, Y) = \bigoplus_R \mathrm{Hom}_{\mathcal{A}}(R \otimes X, Y \otimes R)$.
- $TA = \mathrm{Hom}_{\mathcal{T}(\mathcal{A})}(\bigoplus_R, \bigoplus_S) = \bigoplus_{R,S} \mathrm{Hom}_{\mathcal{A}}(R \otimes S, S \otimes R)$.
- Arrow composition: $g \circ f$ is given by:

$\bigoplus_T \sum_{R,S,b}$ 

The $\mathcal{RT}(\mathcal{A})$ category

Objects of $\mathcal{RT}(\mathcal{A})$: contravariant functors F from $\mathcal{T}(\mathcal{A})$ to Vec.
Example: $\mathrm{Hom}(., Z)$, where Z is a fixed object in $\mathcal{T}(\mathcal{A})$.

$$
\begin{array}{ccc}
\mathcal{T}(\mathcal{A}) & \mathrm{Vec} & \mathrm{Vec} \\
X & F(X) & \mathrm{Hom}(X, Z) \\
u \downarrow & F(u) \uparrow & . \circ u \uparrow \\
Y & F(Y) & \mathrm{Hom}(Y, Z)
\end{array}
$$

F preserves composition of arrows:
$F(u \circ v) = F(v) \circ F(u)$

Arrows of $\mathcal{RT}(\mathcal{A})$: natural transformations ν between functors.

$$
\begin{array}{cccc}
\mathcal{T}(\mathcal{A}) & \mathrm{Vec} & & \mathrm{Vec} \\
X & F(X) & \xrightarrow{\nu_X} & G(X) \\
u \downarrow & F(u) \uparrow & & G(u) \uparrow \\
Y & F(Y) & \xrightarrow{\nu_Y} & G(X)
\end{array}
$$

Notion of equivalence between categories

Categories $\mathcal{A}$ and $\mathcal{B}$ are said to be equivalent if there exists a functor Φ from $\mathcal{A}$ to $\mathcal{B}$ such that:

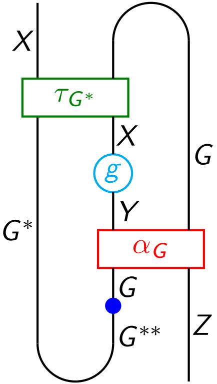
- For all pairs of objects A, A' in $\mathcal{A}$, $\Phi : \mathrm{Hom}_{\mathcal{A}}(A, A') \to \mathrm{Hom}_{\mathcal{B}}(\Phi(A), \Phi(A'))$ is bijective. Φ is said to be fully faithful.
- For any object B in $\mathcal{B}$, there exists an object A in $\mathcal{A}$ such that B is isomorphic to $\Phi(A)$. Φ is said to be essentially surjective.

Equivalence between $Z(\mathcal{A})$ and $\mathcal{RT}(\mathcal{A})$ (I)

Wanted: a functor Φ from $Z(\mathcal{A})$ to $\mathcal{RT}(\mathcal{A})$.
Start from an object (X, τ) in $Z(\mathcal{A})$. We define from it an object $F = \Phi(X, \tau)$ in $\mathcal{RT}(\mathcal{A})$, i.e a functor from $\mathcal{T}(\mathcal{A})$ to Vec.

Action of F on objects in $\mathcal{T}(\mathcal{A})$:
$F(Y) = \mathrm{Hom}_{\mathcal{A}}(Y, X)$.
Action of F on arrows in $\mathcal{T}(\mathcal{A})$:

$\mathcal{T}(\mathcal{A})$	Vec
Z	$F(Z) = \mathrm{Hom}_{\mathcal{A}}(Z, X)$
$\alpha_G \downarrow$	$F(\alpha_G) \uparrow$
Y	$F(Y) = \mathrm{Hom}_{\mathcal{A}}(Y, X)$



Equivalence between $Z(\mathcal{A})$ and $\mathcal{RT}(\mathcal{A})$ (II)

Exercise: Given G, H, R, S, T simple objects in $\mathcal{A}$ and λ, μ so that

$$\lambda \in \mathrm{Hom}_{\mathcal{A}}(G \otimes S, R \otimes G) \quad \text{defines} \quad \lambda_G \in \mathrm{Hom}_{\mathcal{T}(\mathcal{A})}(S, R)$$
$$\mu \in \mathrm{Hom}_{\mathcal{A}}(H \otimes T, S \otimes H) \quad \text{defines} \quad \mu_H \in \mathrm{Hom}_{\mathcal{T}(\mathcal{A})}(T, S)$$

Check that $F(\lambda_G \circ \mu_H) = F(\mu_H) \circ F(\lambda_G)$, i.e. F is a functor from $\mathcal{T}(\mathcal{A})$ to Vec.

Should also be discussed: action of Φ on arrows in $Z(\mathcal{A})$ gives arrows in $\mathcal{RT}(\mathcal{A})$.

For a proof that Φ is an equivalence between $Z(\mathcal{A})$ and $\mathcal{RT}(\mathcal{A})$, see section 7 of Hardiman (2019).

Equivalence between $\mathcal{RT}(\mathcal{A})$ and $\mathrm{Rep}(TA)$

Define $U = \bigoplus S$, object in $\mathcal{T}(\mathcal{A})$. Then: $TA = \mathrm{Hom}_{\mathcal{T}(\mathcal{A})}(U, U)$.
For F object of $\mathcal{RT}(\mathcal{A})$, i.e. a functor from $\mathcal{T}(\mathcal{A})$ to Vec, $F(U)$ is a $\mathbb{C}$ vector space, on which TA acts by right multiplication:
If $f : U \to U \in TA$, $F(f)$ is a linear map $F(U) \to F(U)$.

Notation: for $v \in F(U)$, $F(f)(v) \equiv v.f$, so that:
$F(f \circ g) = F(g) \circ F(f)$ reads $v.(f \circ g) = (v.f).g$

Consider an arrow $\nu : F \longrightarrow G$ in $\mathcal{RT}(\mathcal{A})$:

$$\begin{array}{ccccc}
\mathcal{T}(\mathcal{A}) & & \mathrm{Vec} & & \mathrm{Vec} \\
U & & F(U) & \xrightarrow{\nu_U} & G(U) \\
f \downarrow & & F(f) \uparrow & & G(f) \uparrow \\
U & & F(U) & \xrightarrow{\nu_U} & G(U)
\end{array}$$

commutes,

so ν_U is also an arrow in $\mathrm{Rep}(TA)$.

Equivalence between $\mathcal{RT}(\mathcal{A})$ and $\mathrm{Rep}(TA)$

We have thus defined a functor Ψ from $\mathcal{RT}(\mathcal{A})$ to $\mathrm{Rep}(TA)$.

This is a category equivalence, see **Remark 5.4** of Hardiman (2019). The argument is based on an early result in category theory. See e. g. the book by B. Mitchell, *Theory of categories* (1965), theorem 4.1 page 104.

A glimpse at Morita equivalence

Source: Etingof, Gelaki, Nikshych, Ostrik, *Tensor categories*, in particular sections 7.12 and 7.16.

Consider $\mathcal{M}$ a module category over $\mathcal{C}$.
Define $\mathcal{D} = \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$. It is a tensor category (via composition of functors), with duality (notion of adjoint functor), and $\mathcal{M}$ is also a module category over $\mathcal{D}$.

$\mathcal{C}$ and $\mathcal{D}$ are said to be Morita equivalent. Then $Z(\mathcal{C})$ and $Z(\mathcal{D})$ are equivalent categories.

In particular Vec_G and Rep_G are Morita equivalent, with $\mathcal{M} = Vec$.

What's next?

- Higher genus compact surfaces, ground states and excitations: uses the fact that $Z(\mathcal{A})$ is a modular tensor category, Müger (2003).
- Exact partition function for general string-net models, Ritz-Zwilling, Fuchs, Simon, Vidal, PRB 109, 045130 (2024).
- Aspects of Morita equivalence, Lootens, Vancraeynest-De Cuiper, Schuch, Verstraete, PRB 105, 085130 (2022).
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