Why categories?

- Original framework for mathematical constructions of topological field theories in 2+1 dimensions (Reshetikhin-Turaev (1991), Turaev-Viro (1992)).
- Explicit lattice Hamiltonian formulations (Levin-Wen (2005)).
- Extensions to higher dimensions (next week lecture by C. Delcamp).
- Generalized symmetries and dualities (next week lecture by L. Lootens).
- Provides many new models (general constructions), and also helps to understand why things work: many calculations are replaced by drawings!
Various constructions of TQFT’s in 2+1 dimensions

- **Reshetikhin-Turaev** construction requires a modular tensor category $\mathcal{C}$. Defines a Hilbert space $Z_{RT,\mathcal{C}}(\Sigma)$ for any closed surface $\Sigma$ and a vector $Z_{RT,\mathcal{C}}(\mathcal{M}) \in Z_{RT,\mathcal{C}}(\partial \mathcal{M})$ for any smooth 3-manifold $\mathcal{M}$. This is a non local construction, since it uses surgery of manifolds.

- **Turaev-Viro** (generalized by Barrett-Westbury (1996)) requires a (spherical) fusion category $\mathcal{A}$ as input. This construction is local, and it involves discretized path integrals.

- **Key result:** $Z_{TV,\mathcal{A}} = Z_{RT,Z(\mathcal{A})}$, where $Z(\mathcal{A})$ is the Drinfeld center of $\mathcal{A}$ (Reshetikhin-Virelizier (2010), Balsam-Kirillov (2010)).

- **String nets:** explicit construction of $Z_{TV,\mathcal{A}}(\Sigma)$ as ground-state of a local lattice Hamiltonian (Levin-Wen (2005)). Generalization of Kitaev’s lattice gauge theory model of anyons (1997-2003).
1) Kitaev’s lattice gauge model as a string net: *magnetic* picture
2) String nets from a fusion category
3) Kitaev’s lattice gauge model as a string net: *electric* picture
4) Boundary excitations: the center construction
Consider a planar graph, and a finite group $G$. The Hilbert space of the model is $\mathcal{H} = \mathcal{H}_{ZFC} / \mathcal{N}$. $\mathcal{H}_{ZFC}$ has an orthonormal basis of vectors $|\{g_{ij}\}\rangle$, with $ij$ a link on the lattice, $g_{ij} = g_{ji}^{-1} \in G$, satisfying the zero flux condition: $g_{i_1i_2} g_{i_2i_3} \ldots g_{i_li_1} = e$ for any plaquette bounded by $l$ links.

**Gauge transformations:** Pick $h_i \in G$ for each site $i$. Define $(T_h g)_{ij} = h_i g_{ij} h_j^{-1}$. This transformation preserves the zero flux condition on all plaquettes. $\mathcal{N}$ is the subspace of $\mathcal{H}_{ZFC}$ generated by vectors $|\{g_{ij}\}\rangle - |\{(T_h g)_{ij}\}\rangle$.

$$\mathcal{H}_{ZFC} = \mathcal{H}_{ZFC,S} \oplus \mathcal{N}$$

So $\mathcal{H} = \mathcal{H}_{ZFC} / \mathcal{N} \cong \mathcal{H}_{ZFC,S} =$ ground-state of $(\text{id} - P_S)$. 
2D topological lattice gauge theories (II)

Key fact: On a simply connected planar graph, any fluxless gauge configuration is related to the trivial one \( g_{ij} = e \) by a gauge transformation.

We wish to find \( \{h_i\} \) such that
\[
h_i \, g_{ij} \, h_j^{-1} = e
\]

\[
h_5 = h_0 \, g_{01} \, g_{12} \, g_{23} \, g_{34} \, g_{45}
\]
\[
h_5 = h_0 \, g_{06} \, g_{67} \, g_{78} \, g_{89} \, g_{95}
\]

For a fluxless configuration, both paths give the same \( h_5 \): a non-Abelian and discrete version of Stokes’ theorem. **Cohomological viewpoint** on 2D topological theories.

\( S^2 \) is simply connected, so \( \mathcal{H}(S^2) = \mathbb{C} \). There exists a topological ground-state degeneracy on positive genus closed compact surfaces \( \Sigma \), i.e. \( \dim \mathcal{H}(\Sigma) \geq 2 \rightarrow \) idea of topological quantum computation (Kitaev (1997-2003)).
Sphere with $n$ holes

```

                         0
                        / \
                       /   \
                      /     \ 
k_1                  k_2
                     /   \   /  
                    /     \ /    
                   /  1   2  n  
                  g_1    g_2  g_n

```

fluxless condition through complement of the holes:

\[ k_1 g_1 k_1^{-1} \cdots k_n g_n k_n^{-1} = e \]

Gauge transformations:

\[ k_i \rightarrow h_0 k_i h_i^{-1} \]
\[ g_i \rightarrow h g_i h_i^{-1} \]

Setting $h_i = h_0 k_i$, we get $k_i = e$. So \( \mathcal{H}(S^2, n) = \mathcal{H}_{ZFC}/\mathcal{N} \), where \( \mathcal{H}_{ZFC} \) is spanned by basis vectors \( |g_1, \cdots, g_n\rangle \) such that \( g_1 g_2 \cdots g_n = e \), and \( \mathcal{N} \) is generated by nul vectors \( |g_1, \cdots, g_n\rangle - |h g_1 h^{-1}, \cdots, h g_n h^{-1}\rangle \) associated to gauge transformations.

If $n = 1$, \( \dim(\mathcal{H}(S^2, 1)) \) is equal to the number of conjugacy classes of $G$. For $n \geq 2$, we can fix conjugacy classes \( \text{Cl}_1, \text{Cl}_2, \ldots, \text{Cl}_n \) attached to the holes.

\[ \mathcal{H}(S^2, n, \text{Cl}_1, \ldots, \text{Cl}_n) = \text{Hom}_{\mathcal{Z}(\text{Vec}_G)}(1, (X(\text{Cl}_1), \text{id}) \otimes \cdots \otimes (X(\text{Cl}_n), \text{id})) \]
Anyons from nonsolvable finite groups are sufficient for universal quantum computation

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We present a constructive proof that anyonic magnetic charges with fluxes in a nonsolvable finite group can perform universal quantum computations. The gates are built out of the elementary operations of braiding, fusion, and vacuum pair creation, supplemented by a reservoir of ancillas of known flux. Procedures for building the ancilla reservoir and for correcting leakage are also described. Finally, a universal qudit gate set, which is ideally suited for anyons, is presented. The gate set consists of classical computation supplemented by measurements of the $X$ operator.

FIG. 1. Exchanging two anyons.

FIG. 2. Conjugating a pair of anyons.
Proposed implementation with Josephson circuits

$G$ is the permutation group $S_3$

Douçot, Ioffe, Vidal, PRB 69, 214501 (2005)
Outline

1) Kitaev’s lattice gauge model as a string net: *magnetic* picture
2) String nets from a fusion category
3) Kitaev’s lattice gauge model as a string net: *electric* picture
4) Boundary excitations: the center construction
"Since a category consists of arrows, our subject could also be described as learning how to live without elements, using arrows instead."  S. Mac Lane, Categories for the working mathematician (1971)

Example: Consider an ordered set \((S, \leq)\). It defines a category \(C\), whose objects are elements of \(S\) and \(\text{Hom}(a, b)\) contains a unique arrow if \(a \leq b\), and is empty otherwise.
\( \mathbb{C}\)-linear categories

- \( \text{Hom}(a, b) \) is a finite dimensional vector space over \( \mathbb{C} \), such that composition of arrows is \( \mathbb{C} \)-bilinear:

\[
\begin{align*}
    h \circ (\lambda f + \mu g) &= \lambda (h \circ f) + \mu (h \circ g) & f, g \in \text{Hom}(a, b) \\
    (\lambda h + \mu k) \circ f &= \lambda (h \circ f) + \mu (k \circ g) & h, k \in \text{Hom}(b, c) \\
    \lambda, \mu &\in \mathbb{C}.
\end{align*}
\]

- Existence of a zero object 0, such that \( \text{Hom}(0, 0) = 0 = \{\text{id}_0\} \).

- Existence of direct sums \( a \oplus b \).

\[
\begin{align*}
p \circ i &= \text{id}_a, \quad q \circ j = \text{id}_b \\
q \circ i &= 0, \quad p \circ j = 0 \\
i \circ p + j \circ q &= \text{id}_{a \oplus b}
\end{align*}
\]
Important consequence:

\[ \text{Hom}(\bigoplus_{\alpha} a_{\alpha}, \bigoplus_{\beta} b_{\beta}) \cong \bigoplus_{\alpha, \beta} \text{Hom}(a_{\alpha}, b_{\beta}) \]

\[ f \mapsto \{ f_{\alpha,\beta} = p_{\beta} \circ f \circ i_{\alpha} \} \]

Fusion categories

- Each object \( X \) is a finite direct sum of simple objects \( X_i \):
  \[ X = \bigoplus_i n_i X_i \]  
  \( \text{Hom}(X_i, X_j) = 0 \) if \( i \neq j \) and (\( \mathbb{C} \) alg. closed)  
  \( \text{Hom}(X_i, X_i) = \mathbb{C} \text{id}_{X_i} \).

- There are finitely many simple objects (modulo isomorphisms).
First contact with string net models

Plaquette of lattice gauge model $\rightarrow$ site on the dual lattice. Zero flux condition $g_1 g_2 g_3 g_4 = e$ at each dual lattice site.

The $\text{Vec}_G$ category

- Objects: $G$-graded vector spaces $V = \bigoplus_{g \in G} V_g$ over $\mathbb{C}$
- Arrows from $V$ to $W$: Collection of linear maps $f_g : V_g \rightarrow W_g$
- Simple objects: $\delta_g$ such that $(\delta_g)_h = 0$ if $g \neq h$ and $(\delta_g)_g = \mathbb{C}$.

First step: assign an object of $\text{Vec}_G$ to each link of (dual) lattice. But: how to implement the zero flux condition at (dual) lattice sites?

$g_1 g_2 g_3 g_4 = e \iff \text{Hom}(\mathbb{C}, \delta_{g_1} \otimes \delta_{g_2} \otimes \delta_{g_3} \otimes \delta_{g_4}) \neq 0$

Each site satisfies the $\text{Vec}_G$ fusion rules.
Further requests for $\mathcal{A}$ (input category) from string nets

**Fusion rule** for $\mathcal{A} = \text{Vec}_G$ involves $\text{Hom}(\mathcal{C}, \delta_{g_1} \otimes \delta_{g_2} \otimes \delta_{g_3} \otimes \delta_{g_4})$. For a general $\mathcal{A}$:

- ▶ $\mathcal{C}$ is replaced by a **unit object**, denoted by $\mathbf{1}$.
- ▶ We need a notion of **tensor product**.
  In $\text{Vec}_G$: $(V \otimes W)_g = \bigoplus_h (V_h \otimes W_{h^{-1}g})$.
- ▶ A **link** is adjacent to two sites. We need to swap orientation:
  $g_{ij} \rightarrow g_{ji} = g_{ij}^{-1}$ becomes $V_{ij} \rightarrow V_{ij}^*$: notion of **duality**.
Tensor products: associativity constraints

\[ a \otimes (b \otimes c) \xrightarrow{\alpha_{abc}} (a \otimes b) \otimes c \]

\[ f \otimes (g \otimes h) \]

\[ f \otimes (g \otimes h) \xrightarrow{\alpha_{a'b'c'}} (a' \otimes b') \otimes c' \]

\[ (a \otimes b) \otimes (c \otimes d) \]

\[ a \otimes (b \otimes (c \otimes d)) \]

\[ a \otimes ((b \otimes c) \otimes d) \]

\[ ((a \otimes b) \otimes c) \otimes d \]

\[ ((a \otimes (b \otimes c)) \otimes d \]
Tensor products: the unit object

Mac Lane Coherence theorem: Consider words composed of objects in $\mathcal{A}$, tensor product signs, and parentheses. Pick a pair of words, involving the same sequences of objects, but differing in terms of location of parentheses and of possible occurrences of $\mathbf{1}$. Example: $(((a \otimes \mathbf{1}) \otimes (b \otimes c)) \otimes \mathbf{1}) \otimes d$ and $a \otimes (b \otimes (c \otimes d))$. It is possible to connect them by several different sequences of arrows, involving $\alpha$, $\lambda$ and $\rho$ isomorphisms. Then: all such sequences induce the same arrow between these two words.
Duality (I)

\( V^* \) is a left dual for \( V \) if we have two arrows \( \text{ev}_V : V^* \otimes V \to 1 \) and \( \text{coev}_V : 1 \to V \otimes V^* \) such that (rigidity):

\[
V \overset{\lambda_V^{-1}}{\longrightarrow} 1 \otimes V \overset{\text{coev} \otimes \text{id}}{\longrightarrow} (V \otimes V^*) \otimes V \overset{\alpha^{-1}}{\longrightarrow} V \otimes (V^* \otimes V) \overset{\text{id} \otimes \text{ev}}{\longrightarrow} V \otimes 1 \overset{\rho_V}{\longrightarrow} V = \text{id}_V
\]

\[
V \overset{\rho_{V^*}^{-1}}{\longrightarrow} V^* \otimes 1 \overset{\text{id} \otimes \text{coev}}{\longrightarrow} V^* \otimes (V \otimes V^*) \overset{\alpha}{\longrightarrow} (V^* \otimes V) \otimes V^* \overset{\text{ev} \otimes \text{id}}{\longrightarrow} 1 \otimes V^* \overset{\lambda_{V^*}}{\longrightarrow} V^* = \text{id}_{V^*}.
\]

**Vec**: (finite dimensional vector spaces). \( \text{ev}_V : V^* \otimes V \to \mathbb{C} \) sends \( \varphi \otimes v \) into \( \varphi(v) \). Pick dual bases \( \{\alpha_i\}, \{e_j\} \) for \( V^* \) and \( V \), i.e. \( \alpha_i(e_j) = \delta_{ij} \). \( \text{coev}_V : \mathbb{C} \to V \otimes V^* \) sends \( 1 \in \mathbb{C} \) into \( \sum_i e_i \otimes \alpha_i \).

**Rigidity**: \( v = \sum_i \alpha_i(v) e_i \) and \( \varphi = \sum_i \varphi(e_i) \alpha_i \) for any \( v \in V \) and \( \varphi \in V^* \).

**Vec_G**: \( 1 = \delta_e. \ (V^*)_g = (V_{g^{-1}})^* \).
Graphical representation of duality axiom

\[ V \xrightarrow{\text{coev}} V^* \xrightarrow{\text{ev}} V = V \]

\[ V^* \xrightarrow{\text{coev}} V \xrightarrow{\text{ev}} V^* = V^* \]
Extended coherence theorem: Consider words composed of objects in \( \mathcal{A} \), tensor product signs, and parentheses. Pick a pair of words, differing in terms of location of parentheses and of possible occurrences of \( \mathbf{1} \), but also via possible annihilation (resp. creation) of \( a^*a \) (resp \( aa^* \)) pairs. It is possible to connect them by several different sequences of arrows, involving \( \alpha, \lambda, \rho \) isomorphisms, and \( \text{ev} \) and \( \text{coev} \) arrows. Then: all such sequences induce the same arrow between these two words.

Example:
\[
((a \otimes (b \otimes \mathbf{1})) \otimes c^*) \otimes ((c \otimes d) \otimes \mathbf{1}) \rightarrow (a \otimes e) \otimes ((e^* \otimes b) \otimes d)
\]
\[(a \otimes (b \otimes 1)) \otimes c^\ast) \otimes ((c \otimes d) \otimes 1) \rightarrow (a \otimes e) \otimes ((e^\ast \otimes b) \otimes d)\]
Expression of arrows from simple objects

\[ A = \bigoplus_i n_i X_i, \]  
where \( n_i = \dim \text{Hom}(A, X_i) = \dim \text{Hom}(X_i, A). \]

Consider dual bases \( \{ u_{i\alpha} \} \) for \( \text{Hom}(X_i, A) \), and \( \{ v_{i\alpha} \} \) for \( \text{Hom}(A, X_i) \), i.e. \( v_{i\alpha} \circ u_{i\alpha} = \delta_{ij} \text{id}_{X_i} \). Then:

\[
\text{id}_A = \sum_{i,\alpha} u_{i\alpha} \circ v_{i\alpha}
\]

Consider \( f : A \to B \). \( f = f \circ \text{id}_A = \sum_{i,\alpha} (f \circ u_{i\alpha}^{(A)}) \circ v_{i\alpha}^{(A)}. \)

\[
f \circ u_{i\alpha}^{(A)} = \sum_\mu u_{i\mu}^{(B)} \circ (v_{i\mu}^{(B)} \circ f \circ u_{i\alpha}^{(A)}).
\]

\[
v_{i\mu}^{(B)} \circ f \circ u_{i\alpha}^{(A)} = \langle \mu | F_i^f | \alpha \rangle \text{id}_{X_i}
\]

\[
f \circ u_{i\alpha}^{(A)} = \sum_\mu \langle \mu | F_i^f | \alpha \rangle u_{i\mu}^{(B)}
\]
$A = (X_i \otimes X_j) \otimes X_k, \ B = X_i \otimes (X_j \otimes X_k)$ and $f = a_{X_i,X_j,X_k}$. 

\[
\sum_{r, \mu, \nu} \langle r, \mu, \nu | F_{p}^{ijk} | q, \alpha, \beta \rangle
\]
Definition of $\mathcal{H}_{FR}$

Generalization of $\mathcal{H}_{ZFC}$, defined for $\mathcal{A} = \text{Vec}_G$. Inspired directly by A. Kirillov, Jr., arXiv:1106.6033.

States $|\{g_{ij}\}\rangle$ are replaced by $|\{V(e), \varphi(v)\}\rangle$.

- For each edge $e$ choose an object $V(e)$ in $\mathcal{A}$.
- Arrow reversal: $V(\bar{e}) = V(e)^*$
- For each vertex $v$ choose $\varphi(v) \in \text{Hom}(1, V(e_1) \otimes \ldots \otimes V(e_n))$.

Notion of isomorphism between $\{V(e), \varphi(v)\}$ and $\{V'(e), \varphi'(v)\}$: Defined by a collection of isomorphisms $f_{e_j} : V(e_j) \to V'(e_j)$, such that: $\varphi'(v) = (f_{e_1} \otimes \ldots \otimes f_{e_n}) \circ \varphi(v)$. 

$\vcenter{\begin{tikzpicture}
  \path[-stealth] (0,0) node[below]{$v$} -- (0,1) node[above]{$e_1$} -- (1,2) node[above]{$e_2$} -- (2,1) node[above]{$e_3$} -- (1,0) node[below]{$e_n$};
\end{tikzpicture}}$
Cyclic permutation symmetry around a vertex (I)

\[
\text{Hom}(1, V_1 \otimes ... \otimes V_{n-1} \otimes V_n) \xrightarrow{Z} \text{Hom}(1, V_n \otimes V_1 \otimes ... \otimes V_{n-1})
\]

\[
(f_1 \otimes ... \otimes f_{n-1} \otimes f_n) \circ .
\]

\[
\text{Hom}(1, V'_1 \otimes ... \otimes V'_{n-1} \otimes V'_n) \xrightarrow{Z'} \text{Hom}(1, V'_n \otimes V'_1 \otimes ... \otimes V'_{n-1})
\]

Pivotal structure \(\delta_V : V \rightarrow V^{**}\)
Cyclic permutation symmetry around a vertex: $Z^n = \text{id}$

Consequence of $\delta_{V_1 \otimes V_2} = \delta_{V_1} \otimes \delta_{V_2}$ (set $W = V_1 \otimes \cdots \otimes V_n$)

$\delta$ is a natural transformation rigidity
Goal: define local updates of \( \{ V(e), \varphi(v) \} \), which do not change the state of the system outside of a finite connected region.

How to assign a meaning to this notion?
Definition of the $\mathcal{N}$ subspace (II)

where $\psi$ is given by:
\( \mathcal{H} \) on a sphere

\[ \mathcal{H}(S^2 - \{p\}) = \mathcal{H}(\mathbb{R}^2) = \text{Hom}_\mathcal{A}(1, 1) = \mathbb{C} \]

\( \pi : \mathcal{H}(S^2 - \{p\}) \to \mathcal{H}(S^2) \) surjective so \( \dim \mathcal{H}(S^2) \leq 1 \).

When do we have \( \dim \mathcal{H}(S^2) = 1 \)?

**A:** Constraint near \( p \) should always be satisfied.

\[ \text{tr}_R(f) \]

\[ \text{tr}_L(f) \]

Equality holds when \( \mathcal{A} \) is **spherical**, i.e. when \( \text{tr}_L(f) = \text{tr}_L(f) \) for any arrow \( f \).
Dimension of objects

\[ d_X \delta_X = X \delta_X \]

Kirillov (2011)
The Levin-Wen projector (I)

\[ \frac{1}{D^2} \quad \text{Figure 6. Operator } B_p \]

\[ \mathcal{H}(\Sigma - \{p\}) \xrightarrow{\pi} \mathcal{H}(\Sigma) \]
\[ i \quad \text{Im} B_p \]
\[ \tilde{\pi} \]

\[ B_p \psi - \psi \in \mathcal{N}(\Sigma) \text{ for any } \psi \in \mathcal{H}(\Sigma - \{p\}), \text{ so } \tilde{\pi} \text{ is surjective.} \]

Description of \( \text{Ker } \pi \)

\[ H^{\text{string}}(\Sigma) = H^{\text{string}}(\Sigma - p)/ \langle \langle p \rangle \rangle - \langle \langle p \rangle \rangle \]

Kirillov (2011)

\[ \text{If } \psi \in \text{Ker } \pi \text{ then } B_p \psi = 0, \text{ so } \tilde{\pi} \text{ is injective.} \]
Models for Gapped Boundaries and Domain Walls

Fig. 3. The action of the plaquette operator $B^k_p$: a) the initial state of the plaquette; b) a symbolic representation of the operator $B^k_p$ applied to it; c) the loop is partially fused using Eq. (12) (some labels and the overall factor are not shown); d) the corner triangles have been evaluated to trivalent vertices (summation over $j'_p, \alpha'_q$ is assumed)

1) Kitaev’s lattice gauge model as a string net: *magnetic* picture
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Motivation for $\text{Rep}_G$ string-net

On a given link, associated Hilbert space is $\mathbb{C}[G] = \bigoplus_{g \in G} \mathbb{C} |g\rangle$.

Left action of $G$ on $\mathbb{C}[G]$: $L_h |g\rangle = |hg\rangle$.

Right action of $G$ on $\mathbb{C}[G]$: $R_h |g\rangle = |gh^{-1}\rangle$.

These two actions commute.

\[ \bullet \quad g_{ij} \quad \bullet \]

Gauge transformation: $\mathcal{T}(h_i, h_j) = L_{h_i} \circ R_{h_j}$

Gauge invariant constraint at site $i$: project on singlet subspace of $\bigotimes_j \mathbb{C}[G]_{ij}$ for the $L_{h_i}$ action, equivalent to choose an element in $\text{Hom}_{\text{Rep}_G}(\mathbb{C}, \bigotimes_j \mathbb{C}[G]_{ij})$, i.e. to satisfy $\text{Rep}_G$ fusion rule at site $i$. 
**Basics of $\text{Rep}_G (I)$**

**Objects:** Finite dimensional representations $(E, \rho)$ of $G$.

**Arrows:** $\text{Hom}_{\text{Rep}_G}((E, \rho), (F, \sigma))$ is composed of linear maps $f : E \to F$ such that

\[
\begin{array}{ccccc}
E & \xrightarrow{\rho_g} & E \\
\downarrow f & & & \downarrow f \\
F & \xrightarrow{\sigma_g} & F \\
\end{array}
\]

commutes for any $g$ in $G$.

**Tensor product:** $(E, \rho) \otimes (F, \sigma) \cong (E \otimes F, \rho \otimes \sigma)$.

**Unit object:** $1 = (\mathbb{C}, \text{id})$.

$\text{Hom}_{\text{Rep}_G}(1, (E, \rho)) \cong \{v \in E, \rho_g(v) = v, \forall g \in G\}$: Invariant subspace of $E$ under $\rho$. 

Duality: \((E, \rho)^* = (E^*, \rho^*)\), where \(\rho^*_g = \rho^*_g^{-1}\).

Exercise: Check that \(\text{ev}_E : E^* \otimes E \to \mathbb{C}\) and \(\text{coev}_E : \mathbb{C} \to E \otimes E^*\) defined in Vec also define arrows in \(\text{Rep}_G\).

Simple objects: Finite dimensional irreducible representations \((E_i, \rho_i)\) of \(G\).

Classical decomposition of \(\mathbb{C}[G]\):
As a vector space: \(\mathbb{C}[G] = \bigoplus_i E_i \otimes E_i^*\)
Left action: \(\bigoplus_i (E_i, \rho_i) \otimes (E_i^*, \text{id})\)
Right action: \(\bigoplus_i (E_i, \text{id}) \otimes (E_i^*, \rho_i^*)\)

Used in Buerschaper and Aguado PRB 80, 155136 (2009).
Fluxless constraint in $\text{Rep}_G$ (I)

\[ \mathcal{H} = (\bigotimes_{i=1}^n E_i) \otimes \left( \bigoplus \{g_{ij}\} \mathbb{C} |\{g_{ij}\}\rangle \right) \]

**Rep$_G$ string-net prescription:**
First apply gauge invariance at vertices, to get $\mathcal{H}_{FR}$, using:

\[ T(\{h_i\})(v \otimes |\{g_{ij}\}\rangle) = (\bigotimes_i \rho_i, h_i)(v) \otimes |\{h_i g_{ij} h_j^{-1}\}\rangle \]

Then form $\mathcal{H}_{FR}/\mathcal{N} \cong \text{Hom}_{\text{Rep}_G}(1, \bigotimes_i (E_i, \rho_i))$.

**Question:** Is this equivalent to imposing the fluxless constraint:

\[ g_{12} g_{23} \ldots g_{n-1,n} g_{n1} = e \]
Fluxless constraint in $\text{Rep}_G$ (II)

$$\mathcal{H} = (\bigotimes_{i=1}^{n} E_i) \otimes \left( \bigoplus \{g_{ij}\} \mathbb{C} | \{g_{ij}\} \rangle \right)$$

Fluxless constraint: $g_{12} g_{23} \ldots g_{n-1,n} g_{n1} = e$ defines $\mathcal{H}_{ZFC}$.

Gauge action: $\mathcal{T}(\{h_i\})(v \otimes |\{g_{ij}\}\rangle) = (\bigotimes i \rho_i, h_i)(v) \otimes \{h_i g_{ij} h_j^{-1}\} \rangle$

In fluxless sector, we can bring $\{g_{ij}\}$ to the trivial configuration $\{g_{ij} = e\}$ by a gauge transformation, which has for stabilizer $\{h_i = h\}$, i.e. the diagonal subgroup in $G_1 \times \ldots \times G_n$.

Invariant states in $\mathcal{H}_{ZFC}$ are in 1 to 1 correspondence with invariant states in $\bigotimes_i E_i$ under $\rho = \bigotimes_i \rho_i$, that is $\text{Hom}_{\text{Rep}_G}(1, \bigotimes_i (E_i, \rho_i))$.

This is the expected image subspace of the plaquette projector in the $\text{Rep}_G$ string-net model.
Models for Gapped Boundaries and Domain Walls

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Abstract: We define a class of lattice models for two-dimensional topological phases with boundary such that both the bulk and the boundary excitations are gapped. The bulk part is constructed using a unitary tensor category $\mathcal{C}$ as in the Levin-Wen model, whereas the boundary is associated with a module category over $\mathcal{C}$. We also consider domain walls (or defect lines) between different bulk phases. A domain wall is transparent to bulk excitations if the corresponding unitary tensor categories are Morita equivalent. Defects of higher codimension will also be studied. In summary, we give a dictionary between physical ingredients of lattice models and tensor-categorical notions.

Module categories: associativity constraints

\[
X \otimes (Y \otimes M) \xrightarrow{m_{XYM}} (X \otimes Y) \otimes M
\]

\[
f \otimes (g \otimes h) \quad (f \otimes g) \otimes h
\]

\[
X' \otimes (Y' \otimes M') \xrightarrow{m_{X'Y'M'}} (X' \otimes Y') \otimes M'
\]

\[
(X \otimes Y) \otimes (Z \otimes M)
\]

\[
X \otimes ((Y \otimes (Z \otimes M))) \quad ((X \otimes Y) \otimes Z) \otimes M
\]

\[
X \otimes ((Y \otimes Z) \otimes M) \quad ((X \otimes (Y \otimes Z)) \otimes M
\]

\[
X, Y, Z, X', Y' \text{ in } C \quad M, M' \text{ in } \mathcal{M}
\]
Module categories: behavior of the unit object

Important example of module category:

\( \mathcal{M} = \text{Vec} \) is a module category over \( \mathcal{C} = \text{Vec}_G \) and also over \( \mathcal{D} = \text{Rep}_G \).
Module categories and line defects

Fig. 8. A neighborhood of a defect line between two topological phases, where $i, j, k, l \in C, \lambda_1, \ldots, \lambda_9 \in M, i', j', k', l' \in D$.

Kitaev and Kong (2012)
1) Kitaev’s lattice gauge model as a string net: *magnetic* picture
2) String nets from a fusion category
3) Kitaev’s lattice gauge model as a string net: *electric* picture
4) Boundary excitations: the center construction
Point excitations in string-net models

**Flux**: $\text{Vec}_G$ fusion rule violated at a dual lattice site.

**Charge**: $\text{Rep}_G$ fusion rule violated at an original lattice site.
Definition of $Z(\mathcal{A})(I)$

Objects of $Z(\mathcal{A})$ are pairs $(X, \sigma)$ with $X$ object in $\mathcal{A}$ and $\sigma$ is an half-braiding, i.e. a collection of arrows $\sigma_V : V \otimes X \to X \otimes V$ defined for any object $V$ in $\mathcal{A}$, subject to two conditions:

**Naturality:**

\[
\begin{align*}
V \otimes X & \quad \xrightarrow{\sigma_V} \quad X \otimes V \\
f \otimes \text{id}_X & \quad \xrightarrow{\text{id}_X \otimes f} \quad X \otimes W \\
W \otimes X & \quad \xrightarrow{\sigma_W} \quad X \otimes W
\end{align*}
\]

commutes for all $f \in \text{Hom}_\mathcal{A}(V, W)$.

**Compatibility with tensor product:** $\sigma_{V \otimes W}$ is given by:

\[
\begin{align*}
(V \otimes W) \otimes X & \quad \xrightarrow{\alpha_{V,W,X}} \quad V \otimes (W \otimes X) \xrightarrow{\text{id}_V \otimes \sigma_W} \quad V \otimes (X \otimes W) \\
X \otimes (V \otimes W) & \quad \xrightarrow{\alpha_{X,V,W}} \quad (X \otimes V) \otimes W \xrightarrow{\sigma_V \otimes \text{id}_W} \quad (V \otimes X) \otimes W
\end{align*}
\]
An arrow $f$ of $Z(\mathcal{A})$ from $(X, \sigma)$ to $(Y, \tau)$ is an arrow $f \in \text{Hom}_\mathcal{A}(X, Y)$ such that:

$$V \otimes X \xrightarrow{\sigma_V} X \otimes V$$
$$\text{id}_V \otimes f \xrightarrow{f \otimes \text{id}_V} f \otimes \text{id}_V$$
$$V \otimes Y \xrightarrow{\tau_V} Y \otimes V$$

commutes for any object $V$ in $\mathcal{A}$. 

**Definition of $Z(\mathcal{A})(II)$**

An arrow $f$ of $Z(\mathcal{A})$ from $(X, \sigma)$ to $(Y, \tau)$ is an arrow $f \in \text{Hom}_\mathcal{A}(X, Y)$ such that:
\[ Z(\mathcal{A}) \text{ has tensor products} \]

\[(X, \sigma) \otimes (Y, \tau) = (X \otimes Y, \tau \cdot \sigma), \text{ where } \tau \cdot \sigma \text{ is the half-braiding defined by:} \]

\[
\begin{align*}
V \otimes (X \otimes Y) & \xrightarrow{\alpha_{V,X,Y}^{-1}} (V \otimes X) \otimes Y \xrightarrow{\sigma_V \otimes \text{id}_Y} (X \otimes V) \otimes Y \\
(\tau \cdot \sigma)_V & \xrightarrow{} (X \otimes Y) \otimes V \\
& \xleftarrow{\alpha_{X,Y,V}^{-1}} X \otimes (Y \otimes V) \xrightarrow{\text{id}_X \otimes \tau_V} X \otimes (V \otimes Y)
\end{align*}
\]

Exercises:

- Check that \( \tau \cdot \sigma \) satisfies the two constraints (naturality and compatibility with tensor product) involved in the definition of a half-braiding.

- For \( f : (X, \sigma) \to (X', \sigma'), g : (Y, \tau) \to (Y', \tau') \), check that \( f \otimes g : (X, \sigma) \otimes (Y, \tau) \to (X', \sigma') \otimes (Y', \tau') \) also defines an arrow in \( Z(\mathcal{A}) \).
Associativity of tensor product and unit object in $\mathbb{Z}(\mathcal{A})$

\[
((X, \rho) \otimes (Y, \sigma)) \otimes (Z, \tau) \quad \Rightarrow \quad (X, \rho) \otimes ((Y, \sigma) \otimes (Z, \tau))
\]

\[
((X \otimes Y) \otimes Z, \tau.(\sigma.\rho)) \quad \xrightarrow{\alpha_{X,Y,Z}} \quad (X \otimes (Y \otimes Z), (\tau.\sigma).\rho)
\]

Exercises:

- Check that $\alpha_{X,Y,Z}$ defines an arrow from

  $((X \otimes Y) \otimes Z, \tau.(\sigma.\rho))$ to $(X \otimes (Y \otimes Z), (\tau.\sigma).\rho)$ in $\mathbb{Z}(\mathcal{A})$.

- Check that naturality of $\alpha$ and pentagon identify transfer from $\mathcal{A}$ to $\mathbb{Z}(\mathcal{A})$.

- Check that $(1, \tau)$ with

  $\tau_V = l_V^{-1} \circ r_V : V \otimes 1 \longrightarrow V \longrightarrow 1 \otimes V$ is an object in $\mathbb{Z}(\mathcal{A})$.

- Check that $(1, \tau)$ is a unit object with respect to the tensor product in $\mathbb{Z}(\mathcal{A})$. 
Naturality: \((X, \sigma)\) is determined by a collection of linear maps 
\[ \sigma_{g,k} : X_k \cong (\delta_g \otimes X)_{gk} \rightarrow (X \otimes \delta_g)_{gk} \cong X_{gkg^{-1}} \]

Minimal objects of \(Z(\text{Vec}_G)\) are supported on given conjugacy class \(\text{Cl}(k)\) of \(G\).

Compatibility with tensor product: \(\sigma_{gh,k} = \sigma_{g,hkh^{-1}} \circ \sigma_{g,k}\)

If \(g, h \in \text{Stab}(k)\), \(\sigma_{gh,k} = \sigma_{g,k} \circ \sigma_{g,k}\), so we get a representation \(\rho\) of \(\text{Stab}(k)\), acting on a \(\mathbb{C}\) vector space \(E\).

Description of \((X, \sigma)\): Pick a set of representatives \(\{g_i\}\) so that any element in \(\text{Cl}(k)\) may be uniquely written as \(g_i k g_i^{-1}\). Then:

\[
X_{g_i k g_i^{-1}} = \mathbb{C} e_i \otimes E \\
\sigma_h(e_i \otimes v) = e_j \otimes \rho(s)(v) \\
h g_i = g_j s, \quad s \in \text{Stab}(k)
\]
Description of $Z(Vec_G)$ (II)

Arrows $\hat{f}$ from $(X, \sigma)$ to $(Y, \tau)$:

- If $(X, \sigma)$ and $(Y, \tau)$ are supported on different conjugacy classes: $\text{Hom}_{Z(Vec_G)}((X, \sigma), (Y, \tau)) = 0$.

- If $(X, \sigma)$ and $(Y, \tau)$ are both supported on $\text{Cl}(k)$: $\text{Hom}_{Z(Vec_G)}((X, \sigma), (Y, \tau)) = \text{Hom}_{\text{Rep}(\text{Stab}(k))}(\rho_{\sigma}, \rho_{\tau})$.

$$
\begin{array}{ccc}
E & \xrightarrow{\rho_{\sigma}(s)} & E \\
| & f & | \\
F & \xrightarrow{\rho_{\tau}(s)} & F
\end{array}
$$

commutes for all $s \in \text{Stab}(k)$

$$
\hat{f}(e_i \otimes v) = e_j \otimes f(v).
$$
Magnetic flux excitations correspond to choosing the identity representation of $\text{Stab}(k)$: $E = \mathbb{C}$ and $\text{id}(s) = \text{id}_\mathbb{C}$ for all $s \in \text{Stab}(k)$. The corresponding object

$$X(\text{Cl}(k)) = \bigoplus_{g \in \text{Cl}(k)} \mathbb{C} |g\rangle.$$ Then:

$$\sigma_h(|g\rangle) = |hgh^{-1}\rangle$$

Tensor product of magnetic flux excitations:

$$(X(\text{Cl}_1), \text{id}) \otimes \cdots \otimes (X(\text{Cl}_n), \text{id})$$ is associated to the $G$-graded vector space $X = \bigoplus_{g_i \in C_i} \mathbb{C} |g_1, \ldots, g_n\rangle$. The grading is defined by $|g_1, \ldots, g_n\rangle \in X_{g_1 \ldots g_n}$.

$$\sigma_h |g_1, \ldots, g_n\rangle = |hg_1 h^{-1}, \ldots, hg_n h^{-1}\rangle$$
Motivation: Space of states on a sphere with \( n \) punctures, carrying magnetic flux excitations associated to \( \text{Cl}_1, \cdots, \text{Cl}_n \) conjugacy classes.

Define \( (X, \sigma) = (X(\text{Cl}_1), \text{id}) \otimes \cdots \otimes (X(\text{Cl}_n), \text{id}) \).

\[
\text{Hom}_{\mathbb{Z}(\text{Vec}_G)}(1, (X, \sigma)) = \text{Hom}_{\mathbb{Z}(\text{Vec}_G)}(1, (X_e, \sigma_e)) = \text{Hom}_{\text{Rep}(G)}(\text{id}, \sigma_e) = \{ v \in X_e | \forall h \in G, \sigma_h(v) = v \}
\]

- **Basis for** \( X_e \): \( \{|g_1, \ldots, g_n\} | g_i \in \text{Cl}_i, g_1 \cdots g_n = e \} \).
- \( \sigma_h \) permutes basis vectors.
- **Dimension of invariant vectors subspace** = **number of orbits of basis vectors under** \( \sigma_h \) **permutations** (gauge transformations) = original **lattice gauge theory count**.

\[ \text{Hom}_{\mathbb{Z}(\text{Vec}_G)}(1, (X(\text{Cl}_1), \text{id}) \otimes \cdots \otimes (X(\text{Cl}_n), \text{id})) \]
Equivalence between $Z(\mathcal{A})$ and $\text{Rep}(\mathcal{T}\mathcal{A})$

Developed in Lan and Wen (PRB (2014)). General proof for $\mathcal{A}$ spherical fusion category given by Popa, Shlyakhtenko, Vaes (2018).

Useful, because $\text{Rep}(\mathcal{T}\mathcal{A})$ is semi-simple, i.e. any representation of $\mathcal{T}(\mathcal{A})$ can be decomposed as a direct sum of irreducible representations (Müger (2003)).

I will follow the presentation of Hardiman (arXiv:1911.07271). He introduces a category $\mathcal{T}(\mathcal{A})$ called the tube category of $\mathcal{A}$, and a related category $\mathcal{R}\mathcal{T}(\mathcal{A})$. He shows separately equivalence between $Z(\mathcal{A})$ and $\mathcal{R}\mathcal{T}(\mathcal{A})$ and then between $\mathcal{R}\mathcal{T}(\mathcal{A})$ and $\text{Rep}(\mathcal{T}\mathcal{A})$. 
The tube category $\mathcal{T}(\mathcal{A})$

- **Objects** are the same as the objects of $\mathcal{A}$.
- **Arrows** are different:
  \[
  \text{Hom}_{\mathcal{T}(\mathcal{A})}(X, Y) = \bigoplus_R \text{Hom}_{\mathcal{A}}(R \otimes X, Y \otimes R).
  \]
- $\mathcal{T}A = \text{Hom}_{\mathcal{T}(\mathcal{A})}(\bigoplus_R, \bigoplus_S) = \bigoplus_{R,S} \text{Hom}_{\mathcal{A}}(R \otimes S, S \otimes R)$.
- **Arrow composition**: $g \circ f$ is given by:

\[
\bigoplus_{T \sum_{R, S, b}} \text{b}
\]
The $\mathcal{RT}(\mathcal{A})$ category

**Objects of $\mathcal{RT}(\mathcal{A})$:** contravariant functors $F$ from $\mathcal{T}(\mathcal{A})$ to $\text{Vec}$.  
Example: $\text{Hom}(., Z)$, where $Z$ is a fixed object in $\mathcal{T}(\mathcal{A})$.

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<thead>
<tr>
<th>$\mathcal{T}(\mathcal{A})$</th>
<th>Vec</th>
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<tbody>
<tr>
<td>$X$</td>
<td>$F(X)$</td>
<td>$\text{Hom}(X, Z)$</td>
</tr>
<tr>
<td>$u$</td>
<td>$F(u)$</td>
<td>$\circ u$</td>
</tr>
<tr>
<td>$Y$</td>
<td>$F(Y)$</td>
<td>$\text{Hom}(Y, Z)$</td>
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$F$ preserves composition of arrows: 
$F(u \circ v) = F(v) \circ F(u)$

**Arrows of $\mathcal{RT}(\mathcal{A})$:** natural transformations $\nu$ between functors.

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<th>$\mathcal{T}(\mathcal{A})$</th>
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<tr>
<td>$X$</td>
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<td>$u$</td>
<td>$F(u)$</td>
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<td>$Y$</td>
<td>$F(Y)$</td>
<td>$\nu_Y$</td>
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Categories $\mathcal{A}$ and $\mathcal{B}$ are said to be equivalent if there exists a functor $\Phi$ from $\mathcal{A}$ to $\mathcal{B}$ such that:

- For all pairs of objects $A, A'$ in $\mathcal{A}$, 
  $\Phi : \text{Hom}_\mathcal{A}(A, A') \to \text{Hom}_\mathcal{B}(\Phi(A), \Phi(A'))$ is bijective. $\Phi$ is said to be fully faithful.

- For any object $B$ in $\mathcal{B}$, there exists an object $A$ in $\mathcal{A}$ such that $B$ is isomorphic to $\Phi(A)$. $\Phi$ is said to be essentially surjective.
Equivalence between $Z(\mathcal{A})$ and $\mathcal{RT}(\mathcal{A})$ (I)

**Wanted:** a functor $\Phi$ from $Z(\mathcal{A})$ to $\mathcal{RT}(\mathcal{A})$.

Start from an object $(X, \tau)$ in $Z(\mathcal{A})$. We define from it an object $F = \Phi(X, \tau)$ in $\mathcal{RT}(\mathcal{A})$, i.e. a functor from $\mathcal{T}(\mathcal{A})$ to $\text{Vec}$.

Action of $F$ on objects in $\mathcal{T}(\mathcal{A})$: 
$F(Y) = \text{Hom}_\mathcal{A}(Y, X)$.

Action of $F$ on arrows in $\mathcal{T}(\mathcal{A})$:

\[
\begin{array}{ccc}
\mathcal{T}(\mathcal{A}) & \text{Vec} \\
Z & F(Z) = \text{Hom}_\mathcal{A}(Z, X) \\
\alpha_G & F(\alpha_G) \\
Y & F(Y) = \text{Hom}_\mathcal{A}(Y, X)
\end{array}
\]
Equivalence between $Z(\mathcal{A})$ and $\mathcal{R}\mathcal{T}(\mathcal{A})$ (II)

**Exercise:** Given $G, H, R, S, T$ simple objects in $\mathcal{A}$ and $\lambda, \mu$ so that

$$
\lambda \in \text{Hom}_\mathcal{A}(G \otimes S, R \otimes G) \text{ defines } \lambda_G \in \text{Hom}_{\mathcal{T}(\mathcal{A})}(S, R)
$$

$$
\mu \in \text{Hom}_\mathcal{A}(H \otimes T, S \otimes H) \text{ defines } \mu_H \in \text{Hom}_{\mathcal{T}(\mathcal{A})}(T, S)
$$

Check that $F(\lambda_G \circ \mu_H) = F(\mu_H) \circ F(\lambda_G)$, i.e. $F$ is a functor from $\mathcal{T}(\mathcal{A})$ to $\text{Vec}$.

**Should also be discussed:** action of $\Phi$ on arrows in $Z(\mathcal{A})$ gives arrows in $\mathcal{R}\mathcal{T}(\mathcal{A})$.

For a proof that $\Phi$ is an equivalence between $Z(\mathcal{A})$ and $\mathcal{R}\mathcal{T}(\mathcal{A})$, see section 7 of Hardiman (2019).
Equivalence between $\mathcal{RT}(\mathcal{A})$ and $\text{Rep}(TA)$

Define $U = \bigoplus S$, object in $\mathcal{T}(\mathcal{A})$. Then: $TA = \text{Hom}_{\mathcal{T}(\mathcal{A})}(U, U)$. For $F$ object of $\mathcal{RT}(\mathcal{A})$, i.e. a functor from $\mathcal{T}(\mathcal{A})$ to Vec, $F(U)$ is a $\mathbb{C}$ vector space, on which $TA$ acts by right multiplication:

If $f : U \to U \in TA$, $F(f)$ is a linear map $F(U) \to F(U)$.

Notation: for $v \in F(U)$, $F(f)(v) \equiv v \cdot f$, so that:

$F(f \circ g) = F(g) \circ F(f)$ reads $v \cdot (f \circ g) = (v \cdot f) \cdot g$

Consider an arrow $\nu : F \to G$ in $\mathcal{RT}(\mathcal{A})$:

$$
\begin{array}{ccc}
\mathcal{T}(\mathcal{A}) & \text{Vec} & \text{Vec} \\
U & F(U) & \nu_U \\
\downarrow f & \uparrow F(f) & \uparrow G(f) \\
U & F(U) & \nu_U \\
\end{array}
$$

so $\nu_U$ is also an arrow in $\text{Rep}(TA)$. 
We have thus defined a functor $\Psi$ from $\mathcal{RT}(A)$ to $\text{Rep}(TA)$.

This is a category equivalence, see Remark 5.4 of Hardiman (2019). The argument is based on an early result in category theory. See e. g. the book by B. Mitchell, *Theory of categories* (1965), theorem 4.1 page 104.
A glimpse at Morita equivalence

Source: Etingof, Gelaki, Nikshych, Ostrik, *Tensor categories*, in particular sections 7.12 and 7.16.

Consider $\mathcal{M}$ a module category over $\mathcal{C}$. Define $\mathcal{D} = \text{Fun}_\mathcal{C}(\mathcal{M}, \mathcal{M})$. It is a tensor category (via composition of functors), with duality (notion of adjoint functor), and $\mathcal{M}$ is also a module category over $\mathcal{D}$.

$\mathcal{C}$ and $\mathcal{D}$ are said to be Morita equivalent. Then $Z(\mathcal{C})$ and $Z(\mathcal{D})$ are equivalent categories.

In particular $\text{Vec}_G$ and $\text{Rep}_G$ are Morita equivalent, with $\mathcal{M} = \text{Vec}$. 
What’s next?

- Higher genus compact surfaces, ground states and excitations: uses the fact that $Z(A)$ is a modular tensor category, Müger (2003).
- Exact partition function for general string-net models, Ritz-Zwilling, Fuchs, Simon, Vidal, PRB 109, 045130 (2024).
- Categorical symmetries and dualities, Lootens, Delcamp, Ortiz, Verstraete, PRX Quantum 4, 020357 (2023).